Supersymmetry Theory in Warped Extra Dimension

by

Jingsheng Li

A dissertation submitted to The Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy.

Baltimore, Maryland

Aug, 2013

Abstract

We study supersymmetry theory in higher dimensions. In five dimensional anti-de Sitter space, we construct the most general supersymmetric matter coupling in the rigid and gauged cases.

We use the component and warped $\mathcal{N} = 1$ superspace formalism. By comparing their results, we find several interesting issues related to boundary effects. For instance, Gibbons-Hawking-York terms are necessary on the AdS_5 boundary.

We find the warped space version of spontaneous superpotential generation and reinterpret it as a tuning of a surface-localized Fayet-Iliopoulos term at the component level, or equivalently as a tuning of a boundary-localized superpotential in superspace. On the other hand, in the bulk, we find the $\mathcal{N} = 2$ Fayet-Iliopoulos term must be fixed.

We study boundary problems systematically from the variational principle. We find proper boundary conditions and consistent constraints so that all 8 supercharges are preserved on the boundary. The bulk action is then truly invariant under both supersymmetries; and super-multiplets on the boundary have close relations with the

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superconformal theory in four flat dimensions.

We also investigate some geometric aspects in this thesis. We discover a special type of hyper-Kähler manifold required by the AdS_5 supersymmetry. Such a manifold admits an essential isometry along which two complex structures rotate into each other. We also discuss the complex geometry on the Darboux patch, which plays an important role in hypermultiplets' boundary problems.

Advisor: Prof. Jonathan A. Bagger

Acknowledgments

I would like to express my gratitude to my thesis advisor Jonathan Bagger, without whose patience and continuous support throughout these years, I could never have finished this project. I am deeply indebted to him for his guidance and help.

I want to thank all members in the theoretical group, in particular I thank Prof. David E. Kaplan and Prof. Kirill Melnikov for always filling this group with energy. Prof. Jared Kaplan and Dr. Junpu Wang cleared some of my confusion about AdS/CFT, I thank them for their kindness.

I also thank Prof. Raman Sundrum, for teaching me interesting aspects of AdS and CFT.

I thank all my friends in this department, especially Tom Zorawski, Arpit Gupta, Liang Dai, for sharing an office with me, and providing an environment often with physics discussion. I specially thank Tom for his critical reading of this thesis.

I thank my colleague Dr. Chi Xiong, who also co-worked with me for the early part of this project, for discussions with me and for his suggestions in these years.

I thank Dr. George Bruhn for very useful discussion on Chern-Simons theory.

ACKNOWLEDGMENTS

I thank Prof. Chih-Yung Chien, Prof. Chia-Ling Chien, Prof. Colin Norman for their assistance when I was their TA. And I thank Prof. Oleg Tchernyshyov, Prof. Gabor Domokos and Prof. Susan Kovesi-Domokos for encouragement they gave every time when we met.

I thank my family members for their understanding.

I thank my fiancee Xiaozhou Zhou for everything.

Dedication

This dissertation is dedicated in memory of my late mother, Rongqing Huang, who raised me and taught me to be strong.

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Chapter 1

Introduction

1.1 Motivation

1.1.1 Two Solutions to the "Hierarchy Problem"

The "hierarchy problem" is a puzzle about the mass of the newly discovered Higgs particle. If the Higgs is a fundamental scalar that stays weakly coupled until the energy scale M_{pl}, the quantum correction to its mass-squared m_h^2 will be an enormous number, about $M_{pl}^2 \approx (10^{19}\ GeV)^2$. To have a physical mass-squared around $(127\ GeV)^2$, an unnatural fine-tuning is needed unless new physics shows up at an energy scale much lower than M_{pl}.

Supersymmetry (SUSY) is such a new physics candidate. By enhancing space-time symmetry to supersymmetry, each fermion has a bosonic partner that contributes oppositely to m_h^2. The net effect is a zero contribution, so the hierarchy problem disappears. This simple solution, along with other appealing properties, ("prediction" of gauge coupling unification, providing dark matter candidates, etc.) makes SUSY a favored model. Now that the Higgs has been discovered, the extra particles (a.k.a. superpartners) that weak-scale supersymmetry [1–3] predicts may be the next prey chased down by the Large Hadron Collider and other future hunters.

An alternative solution to the hierarchy problem is to assume the existence of extra dimensions. The basic idea [4] is to dilute 4-d gravity by the extra space volume and to suggest that the true fundamental gravity in the higher dimension becomes strongly coupled around the TeV scale, instead of M_{pl}. One typical model [5] uses one warped extra dimension truncated by two 4-d boundaries. Thus equivalently we can say gravity gets highly "red-shifted" when it reaches us on the boundary.

These two approaches can actually cooperate and help each other. In an extra dimensional model, the distance between two boundaries must be stabilized by some physical mechanism, and SUSY can do it [6, 7]. On the other hand, the Minimal Supersymmetric extension of the Standard Model (MSSM) suffers a serious problem. Since superpartners have yet to be found, SUSY must be broken and all superpartners must be very massive. However, in spontaneous breaking, a sum rule requires that at least some superpaticles have masses lower than their Standard Model partners. This contradicts experimental data. The general solution is to assume a hidden sector outside of the MSSM. Then the sum rule should be applied to both sectors and should not be a severe constraint on the MSSM any more. Extra dimensions provide just such a place to hide the invisible sector. A plausible model [8] can be constructed in the brane-world picture by locating the two sectors on the two separated branes and allowing only gravity to propagate in the bulk.

Thus there is a strong physical motivation for us to investigate supersymmetry in various extra dimensional scenarios. In this thesis we will construct several supersymmetric theories describing interactions among matter and gauge fields. The issue of supersymmetry breaking in these models is left for the future study.

1.1.2 Supersymmetric Nonlinear Sigma Model

We will focus on super-multiplets with spin (0, 1/2) and (0, 1/2, 1). Physically they correspond to matter fields and gauge fields and are called hypermultiplets and vector multiplets respectively. The general interacting theory involving them is the nonlinear sigma model.

Prior to the theory of QCD, the sigma model was introduced as an effective action describing interacting mesons and baryons. It then became a popular tool to study general theories with unbroken and broken symmetries. It turns out that the sigma model has a natural place in supersymmetry too. When one minimizes the supersymmetric action's potential, usually one ends up with some flat directions, i.e. scalars with incompletely constrained arbitrary vacuum expectation values (VEVs). These scalars span a manifold called the moduli space, parametrizing all the possible vacuum configurations. When all the other heavy excitations have been integrated out, the low-energy effective action describing these scalars is just a sigma model with the appropriate amount of supersymmetry. Thus, understanding the sigma model is really one of the fundamental issues in supersymmetry.

Supersymmetry is deeply related to complex geometry. This was first discovered in a sigma model with 4 supercharges: the 2-d $\mathcal{N}$=2 sigma model requires its target space metric to be Kählerian and any Kähler manifold can serve as a supersymmetric sigma model target space [9]. The 4-d $\mathcal{N}$=1 theory has the identical result as the 2-d $\mathcal{N}$=2 one. Already beautiful in a mathematical sense, this result has simple and systematic applications, especially to phenomenological model building.

The case with 8 supercharges was studied by Alvarez-Gaume and Freedman [10] who found that the 2-d $\mathcal{N}$=4 model requires its target space to be hyper-Kählerian. The same result holds in the 4-d $\mathcal{N}$=2 case too. Later, using the $\mathcal{N}$=1 superspace formalism, the discussion was extended to flat 5-d [11] and 6-d [12], where the hypermultiplet's general interactions turned out to be described by hyper-Kähler geometry. All these models live on a flat space-time background.

In this thesis, we systematically construct supersymmetric sigma models in extra dimensional spaces, especially warped ones. The structure of this thesis is as follows: in the rest of this introductory chapter, we will briefly review known results on the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ sigma model in 4 and 5 flat dimensions. In the second chapter, we will construct our rigid sigma model in AdS_5 in the component formalism and then compare with superspace results in the literature. In chapter 3, we will couple the vector multiplet to the hypermultiplet and construct a gauged sigma model in both the component formalism and warped $\mathcal{N}=1$ superspace. Chapter 4 is devoted to the boundary problem. We study the consistent boundary conditions and derive the transformations induced on the boundary value fields. In chapter 5, we make conclusions and comment on potential future work. For self-consistency, notation and some technical details are included in the appendix.

Some material in this thesis has been covered in a publication [13]. Two other research papers on the gauge sigma models and boundary issue will be submitted very soon [14, 15].

1.2 $\mathcal{N}$=1 and $\mathcal{N}$=2 Supersymmetric Nonlinear Sigma Model in 4-dimensions

1.2.1 Supersymmetry Algebra and $\mathcal{N}=1$ Superspace

The $\mathcal{N}=1$ SUSY notation in the thesis follows the textbook by Wess and Bagger [16]; a more detailed discussion can be also found there.

We begin with the minimal ($\mathcal{N}=1$) SUSY algebra in 4 dimensions

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} \;=\; 2\sigma^m_{\alpha\dot\beta} P_m \; . \tag{1.1}$$

A concise way to construct an $\mathcal{N}=1$ invariant model is to use a superspace. Analogous to Minkowski's approach to realize Lorentz transformations as translations and rotations in 3 spatial plus 1 temporal dimensions, supersymmetry transformations can be realized as translations in a superspace with extra Grassmann spinor coordinates θ. Component fields can then be collected and expressed as a single superfield. For instance, in the real superspace $\mathbb{R}^{4|4}$ $(x^m, \theta, \bar\theta)$, matter fields with spin 0 and $\frac{1}{2}$ can be extracted from the following θ-polynomial expansion:

$$\begin{aligned}\Phi(x,\theta,\bar\theta) \;=\;& A(x) + i\theta\sigma^m\bar\theta\partial_m A(x) + \frac{1}{4}\theta^2\bar\theta^2\Box A(x)\\ &\sqrt{2}\theta\chi(x) - \frac{1}{\sqrt{2}} i\theta^2\partial_m\chi(x)\sigma^m\bar\theta\\ &+\theta^2\mathcal{F}(x)\;.\end{aligned} \tag{1.2}$$

The expression can be further simplified in the chiral superspace $\mathbb{C}^{4|2}$ (y^m, θ),

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta\chi(y) + \theta^2\mathcal{F}(y)\;, \tag{1.3}$$

where the ordinary space-time coordinates have been complexified as

$$y^m \equiv x^m + i\theta\sigma^m\bar\theta\;. \tag{1.4}$$

Thanks to this complex extension, Φ becomes manifestly holomorphic:

$$\partial_{\bar\theta}\Phi = 0\;. \tag{1.5}$$

The superfield Φ is called a chiral superfield.

1.2.2 $\mathcal{N}$=1 Supersymmetric Nonlinear Sigma Model

Actions involving $\mathcal{N} = 1$ chiral superfields can be then built up, based on the following observations:

- The $\theta^2\bar{\theta}^2$ component of any general superfield transforms into a total space-time derivative under SUSY.
- The θ^2 component of any chiral superfield transforms into a total space-time derivative under SUSY.
- Extracting components from a superfield is equivalent to integrating this superfield over the Grassmannian variable θ.

For example, the general action describing n chiral multiplets Φ^i only has two terms:

$$S = \int d^4x \left\{ \int d\theta^2 d\bar{\theta}^2 K(\Phi^i, \bar{\Phi}^{j^*}) + \left[\int d\theta^2 P(\Phi^i) + h.c. \right] \right\} , \tag{1.6}$$

where K is a real function with field dependence and P is a holomorphic one. The action is manifestly $\mathcal{N} = 1$ invariant in superspace.

SUSY transformations among components are:

$$\begin{aligned} \delta A^i &= \sqrt{2}\epsilon\chi^i \\ \delta\chi^i &= i\sqrt{2}\sigma^m\bar{\epsilon}\partial_m A^i + \sqrt{2}\epsilon\mathcal{F}^i \\ \delta\mathcal{F}^i &= i\sqrt{2}\bar{\epsilon}\bar{\sigma}^m\partial_m\chi^i \ . \end{aligned} \tag{1.7}$$

Commuting two infinitesimal SUSY transformations generates a space-time translation. For instance

$$\delta_\epsilon \delta_\xi \chi^i - \delta_\xi \delta_\epsilon \chi^i = -2i(\epsilon \sigma^m \bar{\xi} - \xi \sigma^m \bar{\epsilon}) \partial_m \chi^i \ . \tag{1.8}$$

Formulas like this realize the commutation relations (1.1). They are called algebra closure relations. (1.8) is derived without using any equation of motion; supersymmetry is realized off-shell.

Details of the action never show up in off-shell transformations. Such information is hidden in the field $\mathcal{F}$. The advantage of the off-shell formalism is its universal linearity.

However, $\mathcal{F}$ is not a physical field. It has dimension 2 so it never propagates. In fact its Euler-Lagrange equation is an algebraic one and can be solved:

$$\mathcal{F}^i = \frac{1}{2} \Gamma^i_{jk} \chi^j \chi^k - g^{il^*} \nabla_{l^*} \bar{P} \ . \tag{1.9}$$

Integrating out $\mathcal{F}$ then produces the following action involving only physical fields:

$$\begin{aligned} L \ &= \ -g_{ij^*} \partial_m A^i \partial^m A^{*j} - i g_{ij^*} \bar{\chi}^j \bar{\sigma}^m \mathcal{D}_m \chi^i + \frac{1}{4} R_{ij^*kl^*} \chi^i \chi^k \bar{\chi}^j \bar{\chi}^l \\ &\quad -\frac{1}{2} \nabla_i \nabla_j P \chi^i \chi^j - \frac{1}{2} \nabla_{i*} \nabla_{j*} \bar{P} \bar{\chi}^i \bar{\chi}^j - g^{ij^*} \nabla_i P \nabla_{j^*} \bar{P} \ . \end{aligned} \tag{1.10}$$

This action is invariant under the following on-shell SUSY transformation:

$$\begin{cases} \delta_\epsilon A^i = \sqrt{2} \epsilon \chi^i \\ \delta_\epsilon \chi^i = i \sqrt{2} \sigma^\mu \bar{\epsilon} \partial_\mu A^i - \sqrt{2} \Gamma^i_{jk} \delta A^j \chi^k - \sqrt{2} g^{ij^*} \nabla_{j^*} \bar{P} \ , \end{cases} \tag{1.11}$$

where

$$g_{ij^*} \equiv \frac{\partial^2}{\partial A^i \partial A^{j^*}} K(A^i, A^{j^*}) \ . \tag{1.12}$$

The scalars A^i can be viewed as complex coordinates. They span a n-dimensional complex manifold called Kähler manifold. Such manifolds have been well studied in the mathematics community.

In the on-shell formalism, closure on the fermion (1.8) is valid only after using the Dirac equation, and details of the action present as nonlinear terms in the transformations. Both shortcomings can be viewed as advantages though. Because all quantities like g and Γ in the transformation are geometric, building up an ansätz is simple. Since equations of motion are generated in the closure, we can study the algebra closure and construct the action later.

1.2.3 $\mathcal{N}$=2 Supersymmetric Nonlinear Sigma Model in Components

The $\mathcal{N} = 2$ SUSY algebra in 4-d is

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = 2\sigma^m_{\alpha\dot\beta} P_m \delta^{ij} \tag{1.13}$$

$$\{Q^1_\alpha, Q^2_\beta\} = 2\epsilon_{\alpha\beta}\mathcal{Z} \ , \tag{1.14}$$

where $\mathcal{Z}$ is a complex central charge that generates an internal symmetry. $\mathcal{Z}$ commutes with both P and Q.

Analogous to $\mathcal{N} = 1$ superspace, one may expect that by introducing one more Grassmann spinor $\tilde{\theta}$, all $\mathcal{N} = 2$ multiplets can be realized fully off-shell. However, for hypermultiplets this is not true, unless a infinite set of auxiliary fields is used. Examples of superspaces with infinite auxiliary fields are harmonic superspace [17] and projective superspace [18]. Instead, we can realize only one of these two supersymmetries off-shell, using $\mathcal{N} = 1$ superspace. Or one can completely forget about superspace and realize both SUSY on-shell. Taking n hypermultiplets for instance, based on geometric considerations, the on-shell transformation ansätz should have the following form:

$$\left\{ \begin{array}{lcl} \delta A^i & = & \sqrt{2}\Omega^i{}_{k^*}\bar{\eta}\bar{\chi}^k \\ \delta\chi^i_\alpha & = & -i\sqrt{2}\Omega^i{}_{k^*}\sigma^\mu\bar{\eta}\partial_\mu A^{*k} - \sqrt{2}\Gamma^i_{jk}\delta A^j\chi^k - i\sqrt{2}X^i\eta \ . \end{array} \right. \tag{1.15}$$

To match $\mathcal{N} = 1$ transformation (1.11), the holomorphic vector X^i must be

$$X^i = i\Omega^{ij}P_j \ . \tag{1.16}$$

The tensor Ω_{ij} is antisymmetric and holomorphic. Its further properties can be derived from algebra closure:

$$\Omega^i{}_{k^*}\Omega^{k^*}_{\ j} = -\delta^i_j \qquad \text{where} \ \ \Omega^i{}_{k^*} \equiv \Omega^{ij}g_{jk^*}, \ \Omega^{k^*}_{\ j} \equiv \left(\Omega^k{}_{j^*}\right)^* \tag{1.17}$$

$$\nabla_l\Omega^{ij} = \frac{\partial}{\partial A^l}\Omega^{ij} + \Gamma^i_{lp}\Omega^{pj} + \Gamma^j_{lp}\Omega^{ip} = 0 \tag{1.18}$$

$$R_{kl^*pr^*}\Omega^p{}_{m^*}\Omega^{r^*}_{\ s} = R_{kl^*m^*s} \ . \tag{1.19}$$

Condition (1.18) means that the tensor Ω is covariantly constant (metric compatible) and (1.19) is its integrability condition. Using Ω, we can construct 3 complex structures:

$$J^1 = \begin{pmatrix} 0 & -i\Omega^i{}_{j^*} \\ i\Omega^{i^*}{}_j & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & \Omega^i{}_{j^*} \\ \Omega^{i^*}{}_j & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} -i\delta^i_j & 0 \\ 0 & \delta^{i^*}_{j^*} \end{pmatrix} . \tag{1.20}$$

The metric g_{ij^*} is Kählerian with respect to each of these complex structures. Any Kähler manifold with 3 metric compatible complex structures is called a hyper-Kähler manifold. Thus the target space of any $\mathcal{N} = 2$ sigma model is restricted to be hyper-Kählerian.

From the algebra closure, a constraint on X, called the tri-holomorphic condition, can be derived too:

$$\nabla_i X^k + \Omega^{j^*}_{\ i} \nabla_{j^*} \overline{X}^{l^*} \Omega^k{}_{l^*} = 0 . \tag{1.21}$$

After imposing this and using the fermion equations of motion, the closure between the first and the second SUSY reduces to

$$[\delta_\epsilon, \delta_\eta] A^i = 2i(\epsilon\eta - \bar{\epsilon}\bar{\eta}) X^i \tag{1.22}$$

$$[\delta_\epsilon, \delta_\eta] \chi^i = 2i(\epsilon\eta - \bar{\epsilon}\bar{\eta}) \partial_j X^i \chi^j . \tag{1.23}$$

This can be viewed as a diffeomorphism on the hyper-Kähler super-manifold:

$$\begin{cases} \delta_X A^i &= \xi X^i \\ \delta_X \chi^i &= \xi X^i_j \chi^j . \end{cases} \tag{1.24}$$

where the fermions χ^i transform as vectors living on the target space.

The vector X has physical meaning too. According to the commutation relation (1.14), X realizes the central charge symmetry. Thus such a diffeomorphism should be an isometry on the manifold (this corresponds to the invariance of the action), so it must satisfy the Killing equation

$$\delta_X g_{ij^*} = \xi \left(\nabla_i \overline{X}_{j^*} + \nabla_{j^*} X_i \right) = 0 \ . \tag{1.25}$$

X is usually called a tri-holomorphic Killing vector. The geometric meaning of (1.21) is clear: the isometry generated by X preserves the tensor Ω^{ij}:

$$\delta_X \Omega^{ij} = -\xi \Omega^{kj} \left(\nabla_k X^i + \Omega^i{}_{r^*} \nabla_{p^*} \overline{X}^{r^*} \Omega^{p^*}{}_k \right) = 0 \ . \tag{1.26}$$

So along this isometry, the metric g and all three complex structures are invariant.

To summarize, the target space of the 4-d $\mathcal{N}$=2 SUSY sigma-model is a hyper-Kähler manifold admitting a tri-holomorphic Killing vector X.

1.3 $\mathcal{N}$=2 Supersymmetric Nonlinear Sigma Model in Flat 5-d

1.3.1 SUSY Algebra

The generalization to 5-d requires a little bit extra work, mainly due to the spinor notation. In 4-d the minimal irreducible spinor representation is 2-component, while in 5-d it becomes 4-component. In the Dirac 4-component notation, the minimal 5-d SUSY algebra is:

$$\{\mathbb{Q}, \overline{\mathbb{Q}}\} = 2\gamma^M P_M + 2\mathcal{Z} \ , \tag{1.27}$$

where $\overline{\mathbb{Q}} = \mathbb{Q}^\dagger \gamma^0$.

For future $\mathcal{N} = 1$ superspace applications, we can split $\mathbb{Q}$ into 2 Weyl spinors:

$$\mathbb{Q} = \begin{pmatrix} Q_{1\alpha} \\ \bar{Q}_2^{\dot{\beta}} \end{pmatrix} \ , \tag{1.28}$$

the SUSY algebra then takes the following form:

$$\{Q^i_\alpha, \bar{Q}^j_{\dot{\beta}}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^{ij} \tag{1.29}$$

$$\{Q^1_\alpha, Q^2_\beta\} = 2\epsilon_{\alpha\beta}(\mathcal{Z} - iP_5) \ . \tag{1.30}$$

The similarity to the 4-d algebra is obvious. Especially, the 5-d real central charge Z combines with the translation P_5 to form a 4-d complex central charge $\mathcal{Z} = Z - iP_5$. This similarity has a clear interpretation as dimension reduction.

1.3.2 5-d Supersymmetric Nonlinear Sigma Model in 2-component Formalism

The 5-d hypermultiplet can be studied from a 4-d point of view. n hypermultiplets contain $2n$ complex bosons A^i and $2n$ Weyl fermions χ^i. The SUSY transformation ansätz is:

$$\left\{\begin{array}{lcl}\delta A^i &=& \sqrt{2}\epsilon\chi^i+\sqrt{2}\Omega^i{}_{k^*}\bar{\eta}\bar{\chi}^k \\ \delta\chi^i &=& i\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu A^i-i\sqrt{2}\Omega^i{}_{j^*}\overline{X}^{j^*}\epsilon-\sqrt{2}\Omega^i{}_{j^*}\partial_5A^{j^*}\epsilon \\ && -i\sqrt{2}\Omega^i{}_{k^*}\sigma^\mu\bar{\eta}\partial_\mu A^{*k}-i\sqrt{2}X^i\eta+\sqrt{2}\partial_5A^i\eta-\Gamma^i_{jk}\delta A^j\psi^k\ ,\end{array}\right. \tag{1.31}$$

which can be obtained by simply replacing X^i by $X^i-i\partial_5A^i$ in the 4-d transformations.

Constraints on g, Ω and X then can be derived from the algebra closure. The results are identical to the 4-d $\mathcal{N}$=2 case. This is fully expected. To conclude, the most general hypermultiplet in 5-d is described by a sigma model whose target space is a hyper-Kähler manifold with tri-holomorphic isometries.

1.3.3 5-d Supersymmetric Nonlinear Sigma Model in 4-component Formalism

In 5-d, Lorentz rotations rotate Q_1 and $\bar{Q}_2$ into each other. And dynamically, χ^i and $\bar{\chi}^{j^*}$ are mixed by the Dirac equation:

$$i\bar{\sigma}^\mu\mathcal{D}_\mu\chi^i-\mathcal{D}_5(-\Omega^i_{j^*}\bar{\chi}^{j^*})+i\Omega^i_{j^*}\nabla_{r^*}X^{j^*}\bar{\chi}^{r^*}-\frac{1}{2}g^{is^*}R_{jr^*ks^*}(\chi^j\chi^k)\bar{\chi}^{r^*}=0\ . \tag{1.32}$$

To construct an action with manifest 5-d Lorentz invariance, we should work in 4-component spinor formalism. To have simple SUSY transformations, instead of using n unrestricted Dirac spinors, we choose $2n$ restricted ones as

$$\Psi^i=\left(\begin{array}{c}\chi^i \\ -\Omega^i{}_{j^*}\bar{\chi}^{j^*}\end{array}\right)\ . \tag{1.33}$$

In this symplectic Majorana notation, the Dirac equation becomes

$$i\gamma^M \mathcal{D}_M \Psi^i - i\nabla_k X^i \Psi^k + \frac{1}{2} R^i{}_{kp^*l} (\overline{\Psi}^{p^*} \Psi^k) \Psi^l = 0 \ , \tag{1.34}$$

where covariant derivatives on the fermion fields are defined as

$$\mathcal{D}_M \Psi^i = \partial_M \Psi^i + \Gamma^i_{jk} \partial_M A^j \Psi^k \ . \tag{1.35}$$

We then can rewrite the 5-d SUSY transformations as

$$\begin{aligned}
\delta A^i &= \sqrt{2} \bar{\epsilon}^0_+ \Psi^i = \sqrt{2} \Omega^i{}_{j^*} \bar{\Psi}^{j^*} \epsilon^0_- \\
\delta \Psi^i &= \sqrt{2} \left(i\gamma^M \epsilon^0_+ \partial_M A^i - i\Omega^i{}_{j^*} \gamma^M \epsilon^0_- \partial_M A^{*j^*} + iX^i \epsilon^0_+ - i\Omega^i{}_{j^*} \overline{X}^{j^*} \epsilon^0_- \right) \\
&\quad - \Gamma^i_{jk} \delta A^j \Psi^k \ ,
\end{aligned} \tag{1.36}$$

where ϵ^0_+ and its symplectic Majorana dual ϵ^0_- are constant spinors:

$$\epsilon^0_+ = \begin{pmatrix} -\eta \\ \\ \bar{\epsilon} \end{pmatrix}, \qquad \epsilon^0_- = \begin{pmatrix} \epsilon \\ \\ \overline{\eta} \end{pmatrix} \ . \tag{1.37}$$

The SUSY transformations (1.36) preserve the following action:

$$\begin{aligned}
S = \int dx^4 dz \Big\{ & - g_{ij^*} \partial^M A^i \partial_M A^{*j^*} - \mathcal{V}\left(A^i, A^{*j^*}\right) \\
& + \frac{1}{2} i g_{ij^*} \overline{\Psi}^{j^*} \gamma^M \mathcal{D}_M \Psi^i - \frac{1}{8} R_{ij^*kl^*} \left(\overline{\Psi}^{j^*} \Psi^i\right) \left(\overline{\Psi}^{l^*} \Psi^k\right) \\
& - \frac{i}{4} g_{ij^*} \nabla_k X^i \overline{\Psi}^{j^*} \Psi^k + \frac{i}{4} g_{ij^*} \nabla_{l^*} \overline{X}^{j^*} \overline{\Psi}^{l^*} \Psi^i \Big\} \ .
\end{aligned} \tag{1.38}$$

The term $\mathcal{V} = g_{ij^*} X^i X^i$ is called the scalar potential. It contains mass terms, quadratic coupling terms, etc. The scalar potential plays an essential role in the symmetry breaking.

All these results can be viewed as the flat limits ($k \to 0$) of warped space case. In the next chapter, we will find the warped space generalizations of (1.36), (1.37), and (4.21). We will also see whether the tri-holomorphic Killing condition (1.21) changes or not.

Chapter 2

Rigid Supersymmetric Nonlinear Sigma Model in Warped 5-d Space

2.1 AdS$_5$ and Its Isometries

There are several ways to parametrize a warped five dimensional space. To construct an action with manifest 4-d Lorentz invariance, we chose the horospherical coordinates, in which AdS_5 metric is expressed as

$$ds^2 = e^{-2kz}\eta_{\mu\nu}dx^\mu dx^\nu + dz^2 \ . \tag{2.1}$$

On the other hand, AdS_5 can be viewed as the following hypersurface

$$\eta_{AB}Y^A Y^B = \eta_{\mu\nu}Y^\mu Y^\nu + Y^5 Y^5 - Y^6 Y^6 = -\frac{1}{k^2} \ , \tag{2.2}$$

embedded into flat 6 dimensions with signature $(-1,1,1,1,1,-1)$. The embedding metric is

$$ds^2 = \eta_{AB} dY^A dY^B = \eta_{\mu\nu} dY^\mu dY^\nu + dY^5 dY^5 - dY^6 dY^6 \ . \tag{2.3}$$

In these embedding coordinates, it is easy to find all 15 AdS_5 isometry generators[1] as combinations of 6-d Lorentz generators $J_{AB} = Y_A \partial_B - Y_B \partial_A$:

$$\begin{aligned}
P_a &= -ik(J_{a5} + J_{a6}) = -i\delta^\mu_a \partial_\mu && (2.4)\\
M_{ab} &= iJ_{ab} = i\delta_{a\mu}\delta^\nu_b x^\mu \partial_\nu - i\delta^\mu_a \delta_{b\nu} x^\nu \partial_\mu && (2.5)\\
D &= iJ_{56} = ix^\mu \partial_\mu + i\frac{1}{k}\partial_z && (2.6)\\
K_a &= \frac{i}{k}(J_{a5} - J_{a6}) = 2i\delta_{a\mu} x^\mu x^\rho \partial_\rho - i\eta_{\rho\sigma}\delta^\mu_a x^\rho x^\sigma \partial_\mu - \frac{i}{k^2} e^{2kz}\delta^\mu_a \partial_\mu + \frac{2i}{k}\delta_{a\mu} x^\mu \partial_z \ . && (2.7)
\end{aligned}$$

The AdS_5 isometry group is $SO(4,2)$, the same as 4-d conformal group. In (2.4-2.7), names of isometries are chosen to show their one to one correspondence to the 4-d conformal generators. $SO(4,2)$ has commutation relations as follows:

$$\begin{aligned}
&[M_{ab}, M_{cd}] = -2i(\eta_{[ac} M_{b]d} - \eta_{[ad} M_{b]c})\\
&[P_a, M_{bc}] = i\eta_{a[b} P_{c]}, \qquad [K_a, M_{bc}] = i\eta_{a[b} K_{c]},\\
&[P_a, K_b] = -2i(\eta_{ab} D + M_{ab})\\
&[D, P_a] = -iP_a, \qquad [D, K_a] = iK_a \ .
\end{aligned} \tag{2.8}$$

[1] There are alternative ways to realize SO(4,2) isometries. For instance, a 5-d Lorentz covariant choice can be found in [19–21]

2.2 Supersymmetry in AdS$_5$

2.2.1 Supergroup

The supergroup of AdS_5 is SU(2,2|1). Its bosonic set contains not only 15 AdS_5 isometries but also an extra $U(1)$ central charge generator, which can be view as the lift of the R-symmetry in the 4-d superconformal group. The commutation relation is simple in the embedding coordinates[2]:

$$\{\mathbb{Q}, \overline{\mathbb{Q}}\} = -\frac{1}{2}\Sigma^{AB} J_{AB} + 6U \cdot \mathbb{I}\ . \tag{2.9}$$

We can again split Dirac spinor $\mathbb{Q}$ into 2 Weyl ones:

$$\mathbb{Q} = \begin{pmatrix} \frac{1}{\sqrt{k}} Q \\ \sqrt{k}\bar{S} \end{pmatrix}\ , \tag{2.10}$$

then commutation relations become:

$$\begin{aligned}
\{Q_\alpha, \bar{Q}_{\dot\beta}\} &= -2ik\sigma^a_{\alpha\dot\beta}(J_{a5} + J_{a6}) = 2\sigma^a P_a && (2.11)\\
\{S_\alpha, \bar{S}_{\dot\beta}\} &= 2\frac{i}{k}\sigma^a(J_{a5} - J_{a6}) = 2\sigma^a_{\alpha\dot\beta} K_a && (2.12)\\
\{Q_\alpha, S_\beta\} &= -2\sigma^{ab}_{\alpha\beta} J_{ab} + \epsilon_{\alpha\beta}(2J_{56} - 6U) &&\\
&= 2i\sigma^{ab}_{\alpha\beta} M_{ab} - 2i\epsilon_{\alpha\beta} D - 6\epsilon_{\alpha\beta} U\ . && (2.13)
\end{aligned}$$

The central charge U is not really the center of the group $SU(2,2|1)$. In fact it does

[2] 6-d spinor notation is included in App. A.1.

not commute with fermionic charges:

$$[U,Q] = -\frac{1}{2}Q \ , \qquad [U,\bar{Q}] = \frac{1}{2}\bar{Q} \ ,$$
$$[U,S] = \frac{1}{2}S \ , \qquad [U,\bar{S}] = -\frac{1}{2}\bar{S} \ . \tag{2.14}$$

We refer to U as "central charge"[3] when we discuss the bosonic group $SO(4,2)\times U(1)$, and call it "the lift of R-charge in AdS_5" when we discuss supersymmetry.

2.2.2 Killing Spinors

AdS_5 can be viewed as a supersymmetric background solution of 5-d, $\mathcal{N}=2$ gauged supergravity. The SUSY variation of this background must vanish:

$$\delta_\epsilon \Psi_M = \mathcal{D}_M \epsilon - \frac{i}{2}k\gamma_M\epsilon = \partial_M\epsilon + \frac{1}{4}\omega_M^{AB}\gamma_{AB}\epsilon - \frac{i}{2}k\gamma_M\epsilon = 0 \ . \tag{2.15}$$

Thus SUSY transformation parameter ϵ must satisfy the following Killing spinor equation:

$$\mathcal{D}_M\epsilon = \frac{i}{2}k\gamma_M\epsilon \ . \tag{2.16}$$

There is also a simple algebraic reason for ϵ to be a Killing spinor. The algebra closure requires all AdS isometries to be generated by bi-spinor products of ϵ:

$$\{\xi\} = \{\bar{\epsilon}'\epsilon,\ \ \bar{\epsilon}\gamma\epsilon,\ \ \bar{\epsilon}'\gamma\gamma\epsilon\ \ ...\} \ , \tag{2.17}$$

[3]In higher dimension supersymmetry algebras, a "central charges" [22,23] is the generalization of its 4-d version. It may even carry space-time indices so that has non-trivial commutation relations with the Lorentz generators. As long as its reduction corresponds to a 4-d central charge, or R-charge, it is not forbidden by the Coleman-Mandula theorem [24].

The Killing vector equation on ξ then induces the Killing spinor equation (2.16) on ϵ naturally.

We can rewrite (2.16) in the symplectic Majorana notation:

$$\mathcal{D}_M \epsilon_\pm = \pm \frac{i}{2} k \gamma_M \epsilon_\pm \ , \tag{2.18}$$

where ϵ_- is the symplectic Majorana dual of ϵ_+.

There are two independent solutions specified by the constant spinors ϵ and η:

$$\epsilon_+ = \begin{pmatrix} -e^{\frac{1}{2}kz}\eta \\ e^{-\frac{1}{2}kz}\bar{\epsilon} - ike^{-\frac{1}{2}kz}x^\mu \delta^a_\mu \bar{\sigma}_a \eta \end{pmatrix} \ , \tag{2.19}$$

while its symplectic dual takes the following form

$$\epsilon_- = \begin{pmatrix} e^{-\frac{1}{2}kz}\epsilon - ike^{-\frac{1}{2}kz}x^\mu \delta^a_\mu \sigma_a \bar{\eta} \\ e^{\frac{1}{2}kz}\bar{\eta} \end{pmatrix} \ . \tag{2.20}$$

Later we will use these parameters $\epsilon_\pm$ to construct SUSY transformations.

2.3 Rigid Supersymmetric Nonlinear Sigma Model in Components

We are now in position to discuss sigma models in AdS_5. As in flat space, n AdS_5 hypermultiplets contain $2n$ complex scalars A^i and $2n$ Weyl fermions χ^i.

Where appropriate, we collect the fermions into $2n$ symplectic Majorana spinors $\Psi^i = (\chi^i, \Omega^i{}_{j^*}\bar{\chi}^{j^*})^T$ that obey the following constraint

$$\bar{\epsilon}_+ \Psi^i = -\Omega^i{}_{j^*}\bar{\Psi}^{j^*}\epsilon_- \ . \tag{2.21}$$

To find supersymmetry transformations, we write down the most general expressions based on five-dimensional Lorentz covariance, target space diffeomorphism covariance, and the requirement that every slice $z = c$ has $\mathcal{N} = 1$ supersymmetry. That is enough to restrict transformations to be the following form

$$\begin{aligned} \delta A^i &= \sqrt{2}(\epsilon\chi^i + \Omega^i{}_{j^*}\bar{\eta}\bar{\chi}^{j^*}) \\ \delta\chi^i &= \sqrt{2}(i\sigma^m\bar{\epsilon}\partial_m A^i - i\Omega^i{}_{j^*}\sigma^m\bar{\eta}\partial_m A^{*j} - \Omega^i{}_{j^*}\partial_5 A^{j^*}\epsilon + \partial_5 A^i\eta - i\Omega^i{}_{j^*}\bar{X}^{j^*}\epsilon - iX^i\eta) \\ &\quad - \Gamma^i_{jk}\delta A^j\psi^k \ . \end{aligned} \tag{2.22}$$

where the target space is a Kähler manifold. Closure on the bosons tells us that Ω^{ij} must be holomorphic and covariantly constant, so the target-space manifold is also hyper-Kähler. Closure on the fermions implies that X^i is holomorphic and that it satisfies the following constraint:

$$\nabla_j X^i + \Omega^i{}_{p^*}\nabla_{k^*}X^{p^*}\Omega^{k^*}_j = -3ik\delta^i{}_j \ . \tag{2.23}$$

This result differs from the tri-holomorphic condition (1.21) by the nonzero imaginary piece on the right-hand side. For future discussion, we call this condition the inhomogeneous tri-holomorphic condition.

The fermion equations of motion also follow from the closure,

$$i\bar{\sigma}^m \mathcal{D}_m \chi^i + \Omega^i{}_{j^*}\mathcal{D}_5\bar{\chi}^{j^*} - \frac{k}{2}\Omega^i{}_{j^*}\bar{\chi}^{j^*} + i\Omega^i{}_{j^*}\nabla_{k^*}\bar{X}^{j^*}\bar{\chi}^{k^*} - \frac{1}{2}g^{im^*}R_{jk^*lm^*}(\chi^j\chi^l)\bar{\chi}^{k^*} = 0 \ . \tag{2.24}$$

The Killing condition,

$$\nabla_i X_{j^*} + \nabla_{j^*}\bar{X}_i = 0 \ , \tag{2.25}$$

follows from requiring that δ_ϵ and δ_η, acting on (2.24), produce the same bosonic equations of motion. Therefore X^i must be a Killing vector that satisfies the inhomogeneous tri-holomorphic condition (2.23) on the hyper-Kähler manifold. Each sigma model on AdS_5 must carry one such X^i, which we from now on refer to as the essential Killing vector.

In accord with the algebra (2.11), the anti-commutator of $\{Q, S\}$ generates the "central charge" U-transformation:

$$\begin{aligned} \delta_U A^i &= \xi X^i \\ \delta_U \chi^i &= \xi X^i_j \chi^j + \frac{3}{2}ik\xi\chi^i \ , \end{aligned} \tag{2.26}$$

where the parameter $\xi = 2i\,(\epsilon\eta - \bar{\epsilon}\bar{\eta})$. This is an isometry of the hyper-Kähler manifold, and $\delta_U g_{ij^*} = 0$, as required.

It is perhaps more interesting to note that

$$\delta_U \Omega^{ij} = 3ik\xi\Omega^{ij} \ . \tag{2.27}$$

Thus X induces a $O(2)$ rotation between complex structure J^1 and J^2!

$$\begin{pmatrix} \delta_U J^1 \\ \delta_U J^2 \end{pmatrix} = \begin{pmatrix} 0 & 3k\xi \\ -3k\xi & 0 \end{pmatrix} \begin{pmatrix} J^1 \\ J^2 \end{pmatrix} . \tag{2.28}$$

Moreover, the transformation (2.26) is not just the usual diffeomorphism on χ^i, but it includes an additional chiral rotation[4]. In mathematical language, one says that χ^i is a section of a $U(1)$ bundle over the hyper-Kähler manifold.

Given the equations of motion, it is not hard to work backwards to determine the invariant action. We find:

$$\begin{aligned} S \;=\; \int dx^5 \mathrm{e}^{-4kz} \Big\{ & - g_{ij^*}\partial^M A^i \partial_M A^{*j^*} - \mathcal{V}(A^i, A^{*j^*}) \\ & -\frac{i}{2} g_{ij^*} \bar{\Psi}^{j^*} \gamma^M \mathcal{D}_M \Psi^i + \frac{1}{8} R_{ij^*kl^*} (\bar{\Psi}^{j^*}\Psi^i)(\bar{\Psi}^{l^*}\Psi^k) \\ & + \frac{i}{4} g_{ij^*} \nabla_k X^i \bar{\Psi}^{j^*} \Psi^k - \frac{i}{4} g_{ij^*} \nabla_{k^*} \bar{X}^{j^*} \bar{\Psi}^{k^*} \Psi^i \Big\} . \end{aligned} \tag{2.29}$$

where derivatives on fermions are both space-time and target space covariant:

$$\mathcal{D}_M \Psi^i = \partial_M \Psi^i + \frac{1}{4} \omega_M{}^{AB} \gamma_{AB} \Psi^i + \Gamma^i_{jk} \partial_M A^j \Psi^k . \tag{2.30}$$

In the action, the scalar potential $\mathcal{V}$ is now

$$\mathcal{V} = g_{ij^*} X^i \bar{X}^{j^*} - 4k \mathrm{D}^{(X)}(A, A^*) , \tag{2.31}$$

and

$$-i g_{ij^*} \bar{X}^{j^*} = \frac{\partial}{\partial A^i} \mathrm{D}^{(X)} . \tag{2.32}$$

[4]This relates to the fact that U is the lift of $U(1)_R$ symmetry

Equations (2.23) and (2.25) imply that the Killing potential $\mathrm{D}^{(X)}$ is integrable. The action is invariant under supersymmetry when the holomorphic Killing vector X^i satisfies the inhomogeneous tri-holomorphic Killing condition.

For a given hyper-Kähler manifold, one would like to solve (2.23) and (2.25) to find all possible essential Killing vectors X^i. The task is simple when the manifold admits a holomorphic homothetic Killing vector Y^i such that

$$\nabla_j Y^i = -i\delta^i{}_j \ . \tag{2.33}$$

The X^i can then written as

$$X^i = Z^i + \frac{3}{2}kY^i \ , \tag{2.34}$$

where Z is a tri-holomorphic Killing vector that satisfies the usual tri-holomorphic condition. Such manifolds are known as hyper-Kähler cones [25] or Swann spaces [26]. Note that in AdS_5, there is a nonvanishing potential even when $Z^i = 0$.

2.4 Rigid Supersymmetric Nonlinear Sigma Model in Superspace

2.4.1 Warped $\mathcal{N} = 1$ Superspace in AdS$_5$

To connect our component results to the warped $\mathcal{N} = 1$ superspace formalism, we first briefly review this formalism. A systematic study from the co-set construction can be found in [27]. Here we only present the minimal information for our applications.

According to the algebra (2.11), the supergroup of AdS_5 contains two sets of fermionic generators, but only one set satisfies the usual 4-d $\mathcal{N}$=1 anti-commutation relation:

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = \sigma^a_{\alpha\dot{\beta}} P_a \ .$$

The warped $\mathcal{N} = 1$ superspace is a formalism in which Q has the following explicit form:

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\delta^\mu_a \sigma^a_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu \ . \tag{2.35}$$

This $\mathcal{N} = 1$ SUSY transformation acts on chiral multiplets as

$$\delta_\epsilon \Phi(x, \theta, \bar{\theta}) = \left(\epsilon Q + \bar{\epsilon}\bar{Q}\right) \times \Phi(x, \theta, \bar{\theta}) \ . \tag{2.36}$$

Comparing to our component transformations:

$$\delta_\epsilon A^i = e^{-\frac{1}{2}kz}\sqrt{2}\epsilon\chi^i \ , \tag{2.37}$$

we can see the proper collection of component fields is

$$\Phi^i(x, \theta, \bar{\theta}) = A^i(x) + e^{-\frac{1}{2}kz}\sqrt{2}\theta\chi^i(x) + ... \tag{2.38}$$

We can further define covariant differential operators commute with Q_β and $\bar{Q}_{\dot{\beta}}$:

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\delta^\mu_a \sigma^a_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu \ ; \tag{2.39}$$

then, just as in flat 4d, we can define AdS_5 chiral superfields by requiring the following constraint:

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \ . \tag{2.40}$$

Using the new coordinate $y^\mu = x^\mu + i\delta^\mu_a \theta\sigma^a\bar{\theta}$, the chiral superfields containing hypermultiplet components are

$$\Phi^i(y,\theta) = A^i(y) + \sqrt{2}e^{-\frac{1}{2}kz}\theta\chi^i(y) + e^{-kz}\theta^2\mathcal{F}^i(y) \ , \tag{2.41}$$

where auxiliary fields $\mathcal{F}^i$ are introduced to close the first SUSY algebra off-shell.

As far as the first SUSY transformation is concerned, there is no warping information in neither the operator Q nor D. Every warp factor has been absorbed in the superfields. This is the "plain" version of warped superspace. Alternatively, one can absorb the warp factor in the Grassmann θ as well, then operator Q and D will be redefined. In this thesis, we will stick with the "plain" version.

2.4.2 Reduction to the Component Formalism

The most general nonlinear sigma model action in this warped superspace has the following form [27]:

$$S = \int dx^4 dz \Big\{ e^{-2kz} K(\Phi^i, \bar{\Phi}^{j^*})_{\theta^4} + e^{-3kz} \left\{ \left[H_i(\Phi)\partial_5\Phi^i + G(\Phi) \right]_{\theta^2} + h.c. \right\} \Big\} \ . \tag{2.42}$$

Expanding all superfields in components and integrating by parts produces the following action:

$$S = \int dx^4 dz\ e^{-4kz}\Big\{ g_{ij^*}\mathcal{F}^i\mathcal{F}^{*j^*} - \frac{1}{2}g_{im^*}\Gamma^{m^*}_{j^*k^*}\mathcal{F}^i\bar{\chi}^{j^*}\bar{\chi}^{k^*} - \frac{1}{2}g_{mi^*}\Gamma^m_{jk}\mathcal{F}^{*i^*}\chi^j\chi^k$$
$$+ (H_{i,j} - H_{j,i})\mathcal{F}^j\partial_5 A^i + 3kH_i\mathcal{F}^i + G_i\mathcal{F}^i$$
$$+ (\bar{H}_{i^*,j^*} - \bar{H}_{j^*,i^*})\mathcal{F}^{*j^*}\partial_5 A^{*i^*} + 3k\bar{H}_{i^*}\mathcal{F}^{*i^*} + \bar{G}_{i^*}\mathcal{F}^{*i^*}\Big\} + ...$$

Then $\mathcal{F}$ can be solved as follows:

$$\mathcal{F}^i = \frac{1}{2}\Gamma^i_{jk}\chi^j\chi^k - g^{ij^*}(\bar{H}_{k^*,j^*} - \bar{H}_{j^*,k^*})\partial_5 A^{*k^*} - 3kg^{ij^*}\bar{H}_{j^*} - g^{ij^*}\bar{G}_{j^*}\ . \tag{2.43}$$

Using the universal off-shell transformation formula, we find the first SUSY transformation on fermions to be

$$\delta_\epsilon\chi^i = ie^{-\frac{1}{2}kz}\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu A^i + e^{-\frac{1}{2}kz}\sqrt{2}\epsilon\mathcal{F}^i\ . \tag{2.44}$$

Comparing to our fermion transformations

$$\delta_\epsilon\chi^i = \sqrt{2}\,\Big[\ ie^{-\frac{1}{2}kz}\sigma^\mu\bar{\epsilon}\partial_\mu A^i - \Omega^i{}_{j^*}e^{-\frac{1}{2}kz}\epsilon\partial_5 A^{*j^*} - i\Omega^i{}_{j^*}e^{-\frac{1}{2}kz}\epsilon\overline{X}^{j^*}$$
$$+ \frac{1}{2}e^{-\frac{1}{2}kz}\Gamma^i_{jk}(\chi^j\chi^k)\epsilon\ \Big]\ , \tag{2.45}$$

we find

$$\mathcal{F}^i = \frac{1}{2}\Gamma^i_{jk}\chi^j\chi^k - \Omega^i{}_{k^*}\partial_5 A^{*k^*} - i\Omega^i{}_{k^*}\overline{X}^{k^*}\ . \tag{2.46}$$

To match (2.46) with (2.43), the potential H_i in the superspace formalism must connect to Ω_{ij} and X^i in the component formalism:

$$\frac{\partial}{\partial A^i}H_j(A^k) - \frac{\partial}{\partial A^j}H_i(A^k) = \Omega_{ij} \tag{2.47}$$

$$3kH_i(A^k) + G_i(A^k) = -i\Omega_{ij}X^j\ . \tag{2.48}$$

Consistency between these two conditions further requires

$$\Omega_{jk}\nabla_i X^k - \Omega_{ik}\nabla_j X^k = 3ik\Omega_{ij} \ . \tag{2.49}$$

For the Killing vector X^i, this is equivalent to the in-homogenous tri-holomorphic condition (2.23).

In this superspace formalism, one has a freedom to redefine the "potential" H_i as long as it preserves the "field strength" Ω_{ij}. At the action level, this means the superpotential $G(\Phi)$ can be always absorbed into $H_i\partial_5\Phi^i$. In the rest of thesis, we will use this trick to simplify calculations in the warped superspace.

Thus we have matched all component degrees of freedom with the superspace version. The constraints match too. By a tedious though straightforward calculation, we can further match the second SUSY transformations with the expression in [27]. To match the action, especially the scalar potential, the boundary conditions must be taken into account. We leave this problem untill Ch. 4.8.

2.5 Conclusions

In this chapter we constructed a rigid sigma model on a warped 5-d gravitational background called anti-de Sitter space. Using Killing spinors as transformation parameters, on-shell SUSY transformations among component fields were written down. By closing the super algebra $SU(2,2|1)$, all Dirac equations were derived and the SUSY invariant action was constructed from them. Two constraints on the holomorphic vector X was obtained in this approach: a Killing condition and an inhomogenous tri-holomorpihc condition. The second constraint is different from the homogenous version in flat space (both 4-d and 5-d). This makes X an essential Killing vector for any sigma model living on AdS_5. From a geometric point of view, while the usual tri-holomorphic Killing vectors preserve the metric and all 3 complex structures on the hyper-Kähler manifold, the vector X only preserves the metric and the diagonalized complex structure, but rotates the other two into each other.

Comparison to the warped $\mathcal{N} = 1$ superspace formalism confirms our component results: the target space of the non-linear sigma model on AdS_5 is restricted to a class of hyper-Kähler manifolds admitting essential Killing vectors X. An example of such manifold is Swann space, whose homothetic Killing vector can play the role of X. To us, the existence of X on a manifold without homothetic is still unclear yet. We hope mathematicians can help us to find an answer.

Chapter 3

Gauged Supersymmetric Nonlinear Sigma Model in Warped 5-d Space

In this chapter, the rigid sigma model we constructed in Ch. 2 will be gauged. This goal will be archived through two approaches. First we will couple gauge fields to the components fields of hypermultiplets and construct a gauged model with only physical fields. Then in Sec. 3.4, we will use warped $\mathcal{N}$=1 superspace to obtain a more concise action.

3.1 Gauged Supersymmetric Nonlinear Sigma Model in Components

3.1.1 Gauge Multiplet in 5-d

A gauge multiplet in AdS_5 contains one vector v_M, two symplectic Majorana gauginos λ^i and one real gauge scalar Σ. All fields are Lie algebra valued. Without coupling to other multiplets, the on-shell SUSY transformations among the component fields are

$$
\begin{aligned}
\delta v_M^{(a)} &= i\bar{\epsilon}_+ \gamma_M \lambda_+^{(a)} + i\bar{\epsilon}_- \gamma_M \lambda_-^{(a)} && (3.1)\\
\delta \Sigma^{(a)} &= i\bar{\epsilon}_+ \lambda_+^{(a)} + i\bar{\epsilon}_- \lambda_-^{(a)} && (3.2)\\
\delta \lambda_\pm^{(a)} &= \frac{1}{2}\gamma^{MN}\epsilon_\pm F_{MN}^{(a)} + \gamma^M \epsilon_\pm \mathcal{D}_M \Sigma^{(a)} \mp 2ik\epsilon_\pm \Sigma^{(a)} \ , && (3.3)
\end{aligned}
$$

where the gauge covariant derivatives are defined as

$$
\begin{aligned}
& F_{MN}^{(a)} \equiv \mathcal{D}_M v_N - \mathcal{D}_N v_M = \partial_M v_N^{(a)} - \partial_N v_M^{(a)} + g f^{bca} v_M^{(b)} v_N^{(c)} && (3.4)\\
& \mathcal{D}_M \Sigma^{(a)} \equiv \partial_M \Sigma^{(a)} + g f^{bca} v_M^{(b)} \Sigma^{(c)} && (3.5)\\
& \mathfrak{D}_M \lambda_\pm^{(a)} \equiv \partial_M \lambda_\pm^{(a)} + \frac{1}{4}\omega_M^{AB}\gamma_{AB}\lambda_\pm^{(a)} + g f^{bca} v_M^b \lambda_\pm^{(c)} \ . && (3.6)
\end{aligned}
$$

The gauge sector has the following action:

$$
\begin{aligned}
\mathcal{L} \ = \ & -\frac{1}{4}F_{MN}^{(a)}F^{(a)MN} - \frac{1}{2}\mathcal{D}^M\Sigma^{(a)}\mathcal{D}_M\Sigma^{(a)} + 2k^2\Sigma^{(a)2} \\
& -\frac{i}{2}\bar{\lambda}_+^{(a)}\gamma^M\mathfrak{D}_M\lambda_+^{(a)} - \frac{i}{2}\bar{\lambda}_-^{(a)}\gamma^M\mathfrak{D}_M\lambda_-^{(a)} + \frac{k}{4}\bar{\lambda}_+^{(a)}\lambda_+^{(a)} - \frac{k}{4}\bar{\lambda}_-^{(a)}\lambda_-^{(a)} \ .
\end{aligned}
$$

It is invariant under the SUSY transformations (3.1-3.3) and the usual non-Abelian gauge transformations:

$$\delta v_M^{(a)} = \partial_M \alpha^{(a)} + f^{abc} \alpha^{(b)} v_M^{(c)} \tag{3.7}$$

$$\delta \Sigma^{(a)} = f^{abc} \alpha^{(b)} \Sigma^{(c)} \tag{3.8}$$

$$\delta \lambda_\pm^{(a)} = f^{abc} \alpha^{(b)} \lambda_\pm^{(c)} \,. \tag{3.9}$$

In AdS_5, the gauge scalar Σ and the gauginos $\lambda_\pm$ are all massive:

$$|m_{\lambda_\pm}| = \frac{k}{2}, \qquad\qquad m_\Sigma^2 = -4k^2, \tag{3.10}$$

while the gauge vector v_M is still massless (this is necessary for the action to be gauge invariant).

When a gauge multiplet is coupled to hypermultiplets, the on-shell SUSY transformations on fermions from both sectors must be modified, in order to produce the correct EOMs via closure. In the off-shell formalism, all the changes are captured by auxiliary fields, while in the on-shell formalism, we need to covariantize all derivatives in the matter fields' transformations and add functions to the gauginos' transformations as well. These functions should have explicit matter field dependence and produce Yukawa type couplings in the gaugino equation of motion. Based on gauge covariance, the following is the most general ansätz:

$$\begin{aligned}
\delta\lambda_+^{(a)}|_{\text{coupled}} &= \delta\lambda_+^{(a)}|_{\text{uncoupled}} + ig\epsilon_+\mathrm{D}^{(a)} - 2ig\epsilon_-\bar{\mathrm{P}}^{(a)} &(3.11)\\
\delta\lambda_-^{(a)}|_{\text{coupled}} &= \delta\lambda_-^{(a)}|_{\text{uncoupled}} - ig\epsilon_-\mathrm{D}^{(a)} - 2ig\epsilon_+\mathrm{P}^{(a)} &(3.12)\\
\delta\Psi^i|_{\text{coupled}} &= \sqrt{2}\left(i\gamma^M\epsilon_+\mathcal{D}_MA^i + i\Omega^i{}_{j^*}\gamma^M\epsilon_-\mathcal{D}_MA^{*j^*} + iX^i\epsilon_+ + i\Omega^i{}_{j^*}\bar{X}^{j^*}\epsilon_-\right)\\
&\quad -\Gamma^i_{jk}\delta A^j\Psi^k - ig\sqrt{2}\Sigma^{(a)}T^{(a)i}\epsilon_+ - ig\sqrt{2}\Sigma^{(a)}\Omega^i{}_{j^*}\bar{T}^{(a)j^*}\epsilon_- \ , &(3.13)
\end{aligned}$$

where covariant derivatives on the matter fields are defined as

$$\begin{aligned}
\mathcal{D}_MA^i &\equiv \partial_MA^i - gv_M^{(a)}T^{(a)i} &(3.14)\\
\mathfrak{D}_M\Psi^i &\equiv \partial_M\Psi^i + \frac{1}{4}\omega_M^{AB}\gamma^{AB}\Psi^i - gv_M^{(a)}T^{(a)i}{}_j\Psi^j + \Gamma^i_{jk}\mathcal{D}_MA^j\Psi^k \ . &(3.15)
\end{aligned}$$

Consistent closure requires $T^{(a)}$ to be tri-holomorphic Killing vector, i.e.

$$g_{ij^*}\nabla_kT^i + g_{kp^*}\nabla_{j^*}\bar{T}^{p^*} = 0 \qquad (3.16)$$

$$\Omega^i{}_{j^*}\nabla_{k^*}\bar{T}^{j^*}\Omega^{k^*}_{l} + \nabla_lT^i = 0 \ . \qquad (3.17)$$

It also relates the real function $\mathrm{D}^{(a)}(A^i, A^{*j^*})$ and the holomorphic one $\mathrm{P}^{(a)}(A^i)$:

$$\frac{\partial\mathrm{P}^{(a)}}{\partial A^i} = -\Omega^{j^*}_{i}\frac{\partial\mathrm{D}^{(a)}}{\partial A^{*j^*}} \ . \qquad (3.18)$$

The Dirac equations now contain gauge couplings and Yukawa couplings

$$i\gamma^M\mathfrak{D}_M\lambda_+^{(a)} - \frac{k}{2}\lambda_+^{(a)} + igf^{bca}\Sigma^{(b)}\lambda_+^{(c)} - ig\sqrt{2}\mathrm{D}_i^{(a)}\Psi^i = 0 \qquad (3.19)$$

$$i\gamma^M\mathfrak{D}_M\lambda_-^{(a)} + \frac{k}{2}\lambda_-^{(a)} + igf^{bca}\Sigma^{(b)}\lambda_-^{(c)} + ig\sqrt{2}\mathrm{P}_i^{(a)}\Psi^i = 0 \qquad (3.20)$$

$$i\gamma^M\mathfrak{D}_M\Psi^i + \frac{3}{2}k\Psi^i - i\nabla_jX^i\Psi^j + ig\Sigma^{(a)}\nabla_jT^{(a)i}\Psi^j - \sqrt{2}g\left(T^{(a)i}\lambda_+^{(a)} + \Omega^i{}_{j^*}\bar{T}^{(a)j^*}\lambda_-^{(a)}\right) = 0 \ . \qquad (3.21)$$

3.1.2 Minimal Gauged Action

We focus on the minimal coupling scheme, which requires (3.19), (3.20), and (3.21) to be derived from the same action. This further links $\mathrm{D}^{(a)}$, $\mathrm{P}^{(a)}$ to $T^{(a)}$:

$$\mathrm{D}_i^{(a)} \equiv \frac{\partial D}{\partial A^i} = -ig_{ij^*}\bar{T}^{(a)j^*} \tag{3.22}$$

$$\mathrm{P}_i^{(a)} \equiv \frac{\partial P}{\partial A^i} = i\Omega_{ij}T^{(a)j} \; . \tag{3.23}$$

This result shows that $\mathrm{D}^{(a)}$ is the Killing potential and $\mathrm{P}^{(a)}$ is the holomorphic potential for the vector $T^{(a)}$. The tri-holomorphic Killing condition on $T^{(a)}$ is just the integrability condition for $\mathrm{D}^{(a)}$ and $\mathrm{P}^{(a)}$. In general these potentials can only be solved up to integration constants. In non-Abelian case, we can average over the compact group to determine these constants uniquely, so that $\mathrm{D}^{(a)}$ and $\mathrm{P}^{(a)}$ transform homogeneously under the group transformation:

$$T^{(a)i}\mathrm{D}_i^{(b)} + \bar{T}^{(a)j^*}\mathrm{D}_{j^*}^{(b)} = -f^{abc}\mathrm{D}^{(c)} \tag{3.24}$$

$$T^{(a)i}\mathrm{P}_i^{(b)} = -f^{abc}\mathrm{P}^{(c)} \; . \tag{3.25}$$

However, for each Abelian factor in the tri-holomorphic isometry group, these integration constants must be kept as undetermined.

The gauge invariant action can be constructed as:

$$
\begin{aligned}
\mathcal{L} \;=\;& -g_{ij^*}\mathcal{D}^M A^i \mathcal{D}_M A^{j^*} - \mathcal{V}_{\mathcal{G}} \\
& -\frac{i}{2} g_{ij^*} \bar{\Psi}^{j^*} \gamma^M \mathfrak{D}_M \Psi^i + \frac{1}{8} R_{ij^*kl^*} (\bar{\Psi}^{j^*}\Psi^i)(\bar{\Psi}^{l^*}\Psi^k) + \frac{i}{2}\nabla_i X_{j^*} \bar{\Psi}^{j^*}\Psi^i \\
& +\sqrt{2} g T_i^{(a)} \bar{\lambda}_+^{(a)} \Psi^i + \sqrt{2} g T_{j^*}^{(a)} \bar{\Psi}^{j^*} \lambda_+^{(a)} - \frac{i}{2} g \Sigma^{(a)} \nabla_i T_{j^*}^{(a)} \bar{\Psi}^{j^*} \Psi^i \\
& -\frac{1}{4} F_{MN}^{(a)} F^{(a)MN} - \frac{1}{2} \mathcal{D}^M \Sigma^{(a)} \mathcal{D}_M \Sigma^{(a)} + 2k^2 \Sigma^{(a)2} \\
& -\frac{i}{2} \bar{\lambda}_+^{(a)} \gamma^M \mathfrak{D}_M \lambda_+^{(a)} - \frac{i}{2} \bar{\lambda}_-^{(a)} \gamma^M \mathfrak{D}_M \lambda_-^{(a)} + \frac{k}{4} \bar{\lambda}_+^{(a)} \lambda_+^{(a)} - \frac{k}{4} \bar{\lambda}_-^{(a)} \lambda_-^{(a)} \; .
\end{aligned}
\tag{3.26}
$$

The scalar potential now takes the following form

$$
\begin{aligned}
\mathcal{V}_{\mathcal{G}} \;=\;& g_{ij^*} (X^i - g\Sigma^{(a)} T^{(a)i})(\bar{X}^{j^*} - g\Sigma^{(a)} \bar{T}^{(a)j^*}) - 4k\mathrm{D}^{(X)} \\
& +2gk\Sigma^{(a)} \mathrm{D}^{(a)} + \frac{1}{2} g^2 \mathrm{D}^{(a)2} + 2g^2 \mathrm{P}^{(a)} \bar{\mathrm{P}}^{(a)} \; ,
\end{aligned}
\tag{3.27}
$$

where $\mathrm{D}^{(X)}$ is the Killing potential of X^i as given in (2.32).

The gauged action is SUSY invariant when the following constraints are satisfied

$$\Omega_{ij} X^i T^{(a)j} + 3k\mathrm{P}^{(a)} = 0 \tag{3.28}$$

$$X^i T_i^{(a)} - T^{(a)i} X_i = 0 \; . \tag{3.29}$$

When acted upon by derivatives $\partial/\partial A^i$ and $\partial/\partial A^{*j^*}$, both constraints imply the same algebraic condition

$$[X, T^{(a)}] = 0 \; . \tag{3.30}$$

Thus only tri-holomorphic isometries that commute with X can be gauged.

Comparing to the flat space results, we find both (3.28) and (3.29) are stronger.

The flat space case only requires a condition as (3.30). Of course, there X is an usual tri-holomorphic Killing vector instead of the essential one required by AdS_5 SUSY.

The constraint (3.29) has a physical meaning. Prior to being gauged, isometries generated by $T^{(a)}$ must be global symmetries of the rigid sigma model action (2.29) at the first place. When we constructed the gauged action backwards from the algebra closure, this basic requirement has not been checked. It can be shown that for a Killing vector T, the requirement $\delta_T S_{rigid} = 0$ is equivalent to the following condition:

$$\begin{aligned}\delta_T \mathcal{V} &= \delta_T(g_{ij^*} X^i \bar{X}^{j^*}) - 4k\delta_T \mathrm{D}^{(X)} \\ &= [T, X]^i X_i + [\bar{T}, \bar{X}]^{j^*} \bar{X}_{j^*} + X^i \partial_i (T^j X_j - X^j T_j) + 4ik(T^i X_i - X^j T_j) \\ &= 0 \ . \end{aligned} \tag{3.31}$$

In the flat space case ($k = 0$), the last term in $\delta_T \mathcal{V}$ disappears, hence the constraint is solved as $[X, T] = 0$. In App. C.1, we will show this means

$$X^i T_i - T^i X_i = ir \ , \tag{3.32}$$

where r is a real constant.

When $k \neq 0$, requiring the last term in (3.31) to vanish produces $X^i T_i - T^i X_i = 0$, this is the origin of (3.29).

However, this is not the end of the story. Adding boundary terms to the rigid action (2.29) preserves EOMs in the bulk and generates equivalent action. In warped space, an interesting feature is the ambiguity between bulk terms and boundary ones: after integration by parts, a boundary term may be converted to a bulk one. Later in

Sec. 4.8, we will show that adding one particular boundary term a.k.a. Y-term to the action will cancel the last term in the bulk constraint (3.31). The global invariance of the action with Y-term then only requires $[X, T] = 0$ instead of (3.29). In Sec. 4.8 we will also explain why Y-term is necessary. At this moment, we can assume that Y-term is added to the rigid action (2.29), so the only constraint on tri-holomorphic Killing vectors $T^{(a)}$ is (3.28).

(3.28) can be viewed as an equation fixing the holomorphic potential $\mathrm{P}^{(T)}$ completely. As we mentioned, for a compact non-Abelian group, $\mathrm{P}^{(T)}$ has already been fixed by (3.24). It is straightforward to check that the homogenous condition (3.24) is consistent with (3.28). On the other hand, for any Abelian factor, (3.28) is a constraint fixing the integration constant previously we kept as un-determined. This result is new and only valid in the warped space case.

It is interesting to see how the stronger constraint (3.28) turns into the weaker one (3.30) in the $k \to 0$ limit. A naive approach is to take $k = 0$ then (3.28) becomes

$$\Omega_{ij} X^i T^{(a)j} = 0 \ .$$

This is incorrect. The tricky issue is that in the finite k case, $\mathrm{P}^{(T)}$ has an integration constant. This constant can have a part proportional to $1/k$ which survives in the $k \to 0$ limit. Thus the correct flat limit of (3.28) is

$$\Omega_{ij} X^i T^j = c \ , \tag{3.33}$$

where c is a complex constant, and X becomes ordinary tri-holomorphic Killing vector. (3.33) is equivalent to $[X, T] = 0$.

In $\mathcal{N} = 1$ theory, an unfixed constant d in the Killing potential $\mathrm{D}^{(T)}$ corresponds to the Fayet-Iliopoulos (F-I) term. In $\mathcal{N} = 2$ theory, an unfixed constant in the holomorphic potential $\mathrm{P}^{(T)}$ joints $\mathrm{D}^{(T)}$ to form a $\mathcal{N} = 2$ F-I term. So in warped space, gauge invariance and supersymmetry together restrict the P-part of Fayet-Iliopoulos term to a fixed value. Soon, we will see, because of a different reason, the D-part of F-I term is also fixed. As a result, AdS_5 supersymmetry does not allow any free F-I term in the bulk.

Within the essential Killing potential $\mathrm{D}^{(X)}$, there is also a undetermined integration constant. Such a value contributes to the cosmological constant so it must be fixed in order to stabilize the space-time background. We will fix this constant by requiring vanishing of the scalar potential $\mathcal{V}$ when all physical fields are turned off.

Before ending this section, we want to point out the fact that only compact non-Abelian groups should be gauged in the minimal coupling scheme, for the gauge kinetic term $\mathrm{tr}(F^{MN}F_{MN})$ is ghost free only if gauge group $\mathcal{G}$ is compact. On the other hand, when $\mathcal{G}$ is Abelian this restriction is not necessary, non-compact Abelian isometries can be gauged individually in ghost free manner as well.

3.2 Examples

Before continuing more formal discussion, we give a few examples here to illustrate the gauging procedure.

3.2.1 Flat Complex Plane

As the simplest example, let us consider $2n$-d complex plane parameterized by $(z_1, z_2, ...z_{2n})$. We take flat Kähler metric as $g_{ij^*} = \delta_{ij^*}$. The tensor Ω_{ij} has the following standard form:

$$\Omega_{ij} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} . \tag{3.34}$$

The Killing and tri-holomorphic conditions can be written in matrix forms too:

$$\mathrm{T} + \mathrm{T}^\dagger = 0 \tag{3.35}$$

$$\Omega\mathrm{T} + \mathrm{T}^T\Omega = 0 \ , \tag{3.36}$$

where the matrix elements of T are defined as

$$(t)^i_j \equiv \nabla_j T^i = \partial_j T^i \ . \tag{3.37}$$

The homogeneous isometry group now is contained in the unitary group $U(2n)$, while its tri-holomorphic subgroup belongs to $U(2n) \cap Sp(2, \mathbb{C}) = Sp(n)$. These statements are also true in the general hyper-Kähler case, where matrix elements become field dependent and their commutation relations carry additional curvature terms [28].

On the other hand, when the Kähler metric is flat, the integrability condition

$$\nabla_i (t)^j_k - \nabla_k (t)^j_i = 0 \tag{3.38}$$

is satisfied automatically by any constant matrix T. So the homogenous isometry group is the maximal one: $U(2n)$, and the homogenous tri-holomorphic group is $Sp(n)$. The flat complex plane is the maximal symmetric case.

We can choose the essential Killing vector X to be

$$X = -\frac{3}{2} ik \cdot (z_1, z_2, ... z_{2n}) \ , \tag{3.39}$$

thus

$$\mathrm{D}^{(X)} = \frac{3}{2} k \sum_{i=1,..2n} |z_i|^2 \tag{3.40}$$

and the rigid scalar potential is

$$\mathcal{V} = -\frac{15}{4} k^2 \sum_{i=1,..2n} |z_i|^2 \ . \tag{3.41}$$

We can read from this potential that all complex scalars now have the same mass

$$m^2 = -\frac{15}{4} k^2 \ . \tag{3.42}$$

This is called the "conformal" case in the literature [29]. The mass matrix is clearly diagonalized and interactions are absent.

Obviously X commutes with all the $Sp(n)$ generators so we can gauge the whole $Sp(n)$. For instance, when the complex dimension is 2, we can gauge $Sp(1) = SU(2)$

generated by

$$T^{(1)} = \frac{1}{2}(iz_2, iz_1) \tag{3.43}$$

$$T^{(2)} = \frac{1}{2}(z_2, -z_1) \tag{3.44}$$

$$T^{(3)} = \frac{1}{2}(iz_1, -iz_2)\ . \tag{3.45}$$

Their potentials can be solved as

$$\mathrm{P}^{(1)} = (-z_1z_1 + z_2z_2), \quad \mathrm{D}^{(1)} = \frac{-1}{2}(z_1\bar{z}_2 + z_2\bar{z}_1) \tag{3.46}$$

$$\mathrm{P}^{(2)} = -i(z_1z_1 + z_2z_2), \quad \mathrm{D}^{(2)} = \frac{-i}{2}(z_1\bar{z}_2 - z_2\bar{z}_1) \tag{3.47}$$

$$\mathrm{P}^{(3)} = 2z_1z_2, \qquad \mathrm{D}^{(3)} = \frac{-1}{2}(z_1\bar{z}_1 + z_2\bar{z}_2)\ . \tag{3.48}$$

Plugging these into (3.26) will straightforwardly produce the explicit gauged action.

3.2.2 Hyper-Kähler Cone

Since the rigid sigma model on AdS_5 always has a non-zero Killing vector X^i, there is no naive “massless” model. However, when the hyper-Kähler space admits a homothetic Killing vector

$$\nabla_i Y^j = -i\delta_i^j\ , \tag{2.33}$$

Preferred choice of $X^i = \frac{3}{2}kY^i$ is the warped space counterpart of the massless model in flat 5-d. Manifolds that carry homothetic Killing vectors are often called hyper-Kähler cones [25]. As a second example, we demonstrate the gauging procedure on a 2-d hyper-Kähler cone. The complex coordinates are decomposed as (u, z), where u

parameterizes the twistor space $CP(1)$ and z is the coordinate along the homethetic direction. The Kähler metric is

$$g_{ij^*} = \begin{pmatrix} g_{u\bar{u}} & g_{u\bar{z}} \\ g_{z\bar{u}} & g_{z\bar{z}} \end{pmatrix} = \mathrm{e}^{z+\bar{z}} \begin{pmatrix} 1 & \bar{u} \\ u & 1+u\bar{u} \end{pmatrix} , \tag{3.49}$$

while the tenser Ω has the following form:

$$\Omega_{ij} = \begin{pmatrix} \Omega_{uu} & \Omega_{uz} \\ \Omega_{zu} & \Omega_{zz} \end{pmatrix} = \mathrm{e}^{2z} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{3.50}$$

There are 8 independent holomorphic Killing vectors in total:

$$\begin{gathered} Y = (0, -i) \\ N^{(1)} = (\mathrm{e}^{-z}, 0) \qquad N^{(2)} = (i\mathrm{e}^{-z}, 0) \\ N^{(3)} = (-u\mathrm{e}^{-z}, \mathrm{e}^{-z}) \quad N^{(4)} = (-iu\mathrm{e}^{-z}, i\mathrm{e}^{-z}) \\ L^{(1)} = \tfrac{1}{2}(-2iu, i) \qquad L^{(2)} = \tfrac{1}{2}(-u^2-1, u) \quad L^{(3)} = \tfrac{1}{2}(iu^2-i, -iu) \ . \end{gathered} \tag{3.51}$$

Their commutation relations are

$$[Y, L^{(A)}] = 0 \tag{3.52}$$

$$[L^{(A)}, L^{(B)}] = \epsilon^{ABC} L^{(k)} \tag{3.53}$$

$$[N^{(a)}, N^{(b)}] = 0 \tag{3.54}$$

$$[L^{(A)}, N^{(a)}] = -f^{Aab} N^{(b)} \tag{3.55}$$

$$[Y, N^{(a)}] = -f^{Yab} N^{(b)} \ . \tag{3.56}$$

$L^{(A)}$ and $N^{(a)}$ together generate the full tri-holomophic isometry group $\mathcal{G}$. Obviously the set $\{N^{(a)}\}$ forms an Abelian ideal[1]. $\mathcal{G}$ is neither compact nor semi-simple. On the other hand, $\{L^{(A)}\}$ generate a semi-simple subgroup $SU(2)$ of $\mathcal{G}$.

- We may chose the following essential Killing vector to build the rigid model:

$$X^i = \frac{3}{2}kY^i \ , \tag{3.57}$$

The Killing potential is

$$\mathrm{D}^{(X)} = \frac{3}{2}k\mathrm{e}^{z+\bar{z}}(1+u\bar{u}) - \frac{15}{4}k \tag{3.58}$$

and the scalar potential is

$$\mathcal{V} = -\frac{15}{4}k^2\mathrm{e}^{z+\bar{z}}(1+u\bar{u}) + \frac{15}{4}k^2 \ . \tag{3.59}$$

This is the "massless" model.

In this case, because $[X, L^{(A)}] = 0$, the full $SU(2)$ group can be gauged. The gauge potentials can be solved as

$$\mathrm{P}^{(1)} = -2\mathrm{e}^{2z}u, \qquad \mathrm{D}^{(1)} = \frac{1}{2}\mathrm{e}^{z+\bar{z}}(u\bar{u}-1) \tag{3.60}$$

$$\mathrm{P}^{(2)} = -\mathrm{e}^{2z}\left(-iu^2 - i\right), \quad \mathrm{D}^{(2)} = \frac{i}{4}\mathrm{e}^{z+\bar{z}}(u-\bar{u}) \tag{3.61}$$

$$\mathrm{P}^{(3)} = -\mathrm{e}^{2z}\left(1-u^2\right), \quad \mathrm{D}^{(3)} = \frac{1}{4}\mathrm{e}^{z+\bar{z}}(u+\bar{u}) \ . \tag{3.62}$$

- We can make other choice of X to build the rigid model. For instance,

$$X = \frac{3}{2}k\left(Y + 2L^{(1)}\right) \ , \tag{3.63}$$

[1] Because $N^{(a)j}\nabla_j N^{(a)i} = 0$, geometrically speaking, $N^{(a)}$ are local geodesics directions that generate translations.

thus

$$\mathrm{D}^{(X)} = 3k\mathrm{e}^{z+\bar{z}} - 3k \tag{3.64}$$

and the scalar potential in this case becomes

$$\mathcal{V} = 9k^2 u\bar{u}\ \mathrm{e}^{z+\bar{z}} - 12k^2\left(\mathrm{e}^{z+\bar{z}} - 1\right)\ . \tag{3.65}$$

Physically, this means the rigid model is manifestly massive now. In this case, the tri-holomorphic Killing vectors commuting with X^i are $\{L^{(1)}, N^{(1)}, N^{(2)}\}$. They generate a non-compact Lie group $ISO(2)$. According to our general statement, only individual one-parameter Lie groups (1 compact $U(1)$, 2 non-compact $\mathbb{R}^1$ and their linear combinations) can be gauged respectively. Their potentials can be determined as

$$\mathrm{G} = -12ke^{z+\bar{z}} \tag{3.66}$$

$$\mathrm{P}^{(L1)} = -2\mathrm{e}^{2z}u, \quad \mathrm{D}^{(L1)} = \frac{1}{2}\mathrm{e}^{z+\bar{z}}(u\bar{u} - 1) + d_1 \tag{3.67}$$

$$\mathrm{P}^{(N1)} = -4i\mathrm{e}^{z}, \quad \mathrm{D}^{(N1)} = iue^{z} - i\bar{u}e^{\bar{z}} + d_2 \tag{3.68}$$

$$\mathrm{P}^{(N2)} = 4\mathrm{e}^{z}, \quad \mathrm{D}^{(N2)} = ue^{z} + \bar{u}e^{\bar{z}} + d_3\ , \tag{3.69}$$

where the undetermined d_i here are real constants that correspond to $\mathcal{N} = 1$ Fayet-Iliopoulos parameters to be fixed in the next section.

3.3 Generation of Superpotential

When space-time is flat, there is a very interesting fact in $\mathcal{N}=2$ model. It was first illustrated by Fayet in 4-d [30, 31], then by Hull et. al. in 5-d [32].

Taking Abelian gauge theory for instance. First, notice a shift symmetry in the free gauge section:

$$\Sigma \rightarrow \Sigma + \sigma \ . \tag{3.70}$$

In flat space, gauge fields and their super partners are massless, so shifting the gauge scalar by a constant obviously preserves the gauge sector action. Furthermore, since the Σ only appear in the SUSY transformation with derivatives, a constant shift of Σ preserves supersymmetry.

Next consider the shift (3.70) as a field redefinition in the coupled theory. Physically such a shift corresponds to assigning a VEV to the scalar field Σ. It turns out that a superpotential is then generated in the matter sector. We can clearly see this from the scalar potential with $X^i = 0$.

$$\begin{aligned} \mathcal{V}_{\mathcal{G}} \quad &= \quad g_{ij^*}(-g\Sigma T^i)(-g\Sigma \bar{T}^{j^*}) \\ &\longrightarrow \quad g_{ij^*}(\tilde{X}^i - g\Sigma T^i)(\bar{\tilde{X}}^{j^*} - g\Sigma \bar{T}^{j^*}) \ , \end{aligned} \tag{3.71}$$

where the generated Killing vector $\tilde{X}^i = -g\sigma T^i$.

The non-Abelian version is more interesting. Since the kinetic term of $\Sigma^{(a)}$ contains a $g^2 f^{abc} f^{ade} \Sigma^{(c)} \Sigma^{(e)} v_M^{(b)} v_M^{(d)}$ piece, shifting some $\Sigma^{(a)}$ will create mass terms for some gauge vectors $v_M^{(b)}$. More explicitly, the gauge symmetry will be spontaneously broken, and the unbroken $U(1)$ generator $Z = \sigma^{(a)}T^{(a)}$ will commute with all unbroken Yang-Mills generators and will show up as the central charge in the supersymmetric algebra. Hence the mechanism is also known as spontaneously generation of central charge [31]. As Hull et. al. pointed out [32], it relates to the non-zero $\mathcal{N} = 2$ Fayet-Iliopoulos terms.

We want to study the warped version of this mechanism. Immediately, in the gauge sector we find that such a shift generates a non-linear transformation on gaugino, as

$$\delta\lambda_{\pm}^{(a)} = ... \mp 2ik\epsilon_{\pm}\sigma^{(a)} \ . \tag{3.72}$$

Thus a constant shift on Σ appears to break both the first and the second SUSY spontaneously in AdS_5.

A more severe problem is present: the shifted vacuum is not stable! The mass term of Σ on AdS_5 generates a linear term under (3.70):

$$2k^2\Sigma^2 \to 2k^2\Sigma^2 + 2k^2\sigma^2 + 4k^2\sigma\Sigma \ . \tag{3.73}$$

So in warped space, shifting the gauge scalar by a constant is neither a symmetry of the gauge sector action nor a legal redefinition.

When gauge multiplets couple to hypermultiplets, there is an interesting solution to both problems. In the gauged scalar potential, the term $-2gk\Sigma\mathrm{D}$ allows us to perform a compensating shift on the function D to cancel the linear mass term in (3.73):

$$\mathrm{D} \to \mathrm{D} + d \ , \tag{3.74}$$

where

$$d = \frac{2k}{g}\sigma \ . \tag{3.75}$$

These combined shifts produce

$$\begin{aligned} & 2k^2\Sigma^2 - 2gk\Sigma\mathrm{D} - \frac{1}{2}g^2\mathrm{D}^2 + 4k\mathrm{D}^{(X)} \\ \longrightarrow \quad & 2k^2\Sigma^2 - 2gk\Sigma\mathrm{D} - \frac{1}{2}g^2\mathrm{D}^2 + 4k\left(\mathrm{D}^{(X)} - g\sigma\mathrm{D} - \frac{k}{2}\sigma^2\right) \ . \end{aligned} \tag{3.76}$$

Furthermore, the compensating shift also restore the linear supersymmetry transformation:

$$\delta\lambda_\pm = ... \mp 2ik\epsilon_\pm\Sigma \pm ig\epsilon_\pm\mathrm{D} \longrightarrow \delta\lambda_\pm \ . \tag{3.77}$$

As a result, the action is stabilized, while the SUSY is unbroken as well. We kill two birds with a single stone.

(3.75) has an important physical implication. In order to stabilize the vacuum, the integration constant of D must be chosen to satisfy (3.75). Therefore, in any given physical vacuum, AdS_5 Fayet-Illiopoulos terms are completely fixed by (3.75)! The analogous $\mathcal{N}=1$ result has also been recently found in warped 4-d [33] using a different approach.

In the non-Abelian case, since we have no freedom to shift $\mathrm{D}^{(a)}$, Fayet's mechanism does not exist at all. In Abelian case, D has an undetermined integration constant, the combined shift (3.74) is then allowed; and the constant is actually fixed by

$$< D >= \frac{2k}{g} < \Sigma > \ . \tag{3.78}$$

This simple mechanism can be useful for model building and for investigating SUSY breaking in AdS_5. If we view values of $< \Sigma >$ and $< \mathrm{D} >$ parametrize the set of all allowed sigma models, Fayet's mechanism will move us from one model to another one. According to (3.76), the new model has a new essential Killing vector as

$$X^i_{\mathrm{new}} = X^i_{\mathrm{old}} - g\sigma T^i \ , \tag{3.79}$$

while the new Killing potential is:

$$\mathrm{D}^{(X)}_{\mathrm{new}} = \mathrm{D}^{(X)}_{\mathrm{old}} - g\sigma \mathrm{D}^{(T)} - \frac{k}{2}\sigma^2 \ . \tag{3.80}$$

The constant in $\mathrm{D}^{(X)}_{\mathrm{new}}$ is removable. A much clearer point of view will be presented when we use $\mathcal{N} = 1$ superspace to study this issue later.

3.4 Gauged Supersymmetric Nonlinear Sigma Model in Superspace

Now we switch to warped $\mathcal{N} = 1$ superspace to reconstruct the gauged nonlinear sigma model using superfields. We will start with the easier case with Abelian gauge symmetry, where gauge transformations have a simple linear form. After constructing the $U(1)$ gauged model, we will use it to investigate the spontaneous superpential generation issue. Generalization to the non-Abelian symmetry is archived in Sec. 3.4.5

3.4.1 $U(1)$ Gauge Multiplet in Warped $\mathcal{N}=1$ Superspace

We first realize the gauge multiplet off-shell. The components of a AdS_5 gauge multiplet can be collected into two $\mathcal{N}=1$ superfields: a real one $V(x,\theta,\bar{\theta})$ and a chiral one $\chi(y,\theta)$. On $U(1)$ gauge multiplet, gauge transformations have the following form:

$$\delta_\Lambda V = i\Lambda - i\bar{\Lambda} \tag{3.81}$$

$$\delta_\Lambda \chi = i\partial_5 \Lambda \ . \tag{3.82}$$

The gauge freedom allows us to fix V in the Wess-Zumino gauge. Then the component expansion of superfields become:

$$V = -\theta\sigma^a\bar{\theta}\delta_a^m v_m + i\bar{\theta}^2\theta \mathrm{e}^{-\frac{3}{2}kz}\lambda_1 + i\theta^2\bar{\theta}\mathrm{e}^{-\frac{3}{2}kz}\bar{\lambda}_1 + \frac{1}{2}\theta^2\bar{\theta}^2\mathrm{e}^{-2kz}D \tag{3.83}$$

$$\chi = \frac{1}{2}(\Sigma + iv_5) + i\theta\mathrm{e}^{-\frac{kz}{2}}\lambda_2 + \theta^2 e^{-kz}F \ , \tag{3.84}$$

where D and F are auxiliary fields in the gauge multiplet. To distinguish them from other similar symbols, in this thesis we always use the straight roman typeset D for the Killing potential, F for the holomorphic potential, and the curly letter $\mathcal{F}$ for auxiliary fields in hypermultiplets.

The invariant action for gauge sector reads

$$S = \int d^5x \left\{ \int d^4\theta \mathrm{e}^{-2kz}\frac{1}{g^2}(\chi+\bar{\chi}-\partial_5 V)^2 + \left[\int d^2\theta \frac{1}{4g^2}W^\alpha W_\alpha + h.c.\right]\right\} \ . \tag{3.85}$$

Using warped $\mathcal{N}=1$ superspace, the invariance under the first SUSY is manifest; and the second SUSY transformation preserving the action (3.83) has the following form:

$$\delta_\eta V = 2(\chi + \bar{\chi} - \partial_5 V)(\theta\eta + \bar{\theta}\bar{\eta}) - k\eta_s^A D_A V \tag{3.86}$$

$$\delta_\eta \chi = -e^{2kz}\eta W - k\eta_s^A D_A \chi \ , \tag{3.87}$$

where $\eta_s^A = (\eta_s^a, \eta_s^\alpha, \eta_{s\dot{\alpha}})$ is a set of superfields with the following expansions [34]:

$$\eta_s^a = -2(\theta\sigma^a\bar{\sigma}^b\eta + \bar{\eta}\bar{\sigma}^b\sigma^a\bar{\theta})\delta_{mb}x^m + 2i\bar{\theta}^2\theta\sigma^a\bar{\eta} + \theta^2\bar{\theta}\sigma^a\eta \tag{3.88}$$

$$\eta_s^\alpha = -i(\bar{\eta}\bar{\sigma}^a)^\alpha\delta_{ma}x^m + 2(\bar{\theta}\bar{\eta})\theta^\alpha + 2\theta^2\eta^\alpha \ . \tag{3.89}$$

Without using any equation of motion, it can be shown that all commutators $[\delta_\epsilon, \delta_{\epsilon'}]$, $[\delta_\eta, \delta_{\eta'}]$, $[\delta_\epsilon, \delta_\eta]$ acting on superfields generate AdS_5 isometries, the central charge symmetry, and a gauge symmetry as well. SUSY is closed off-shell for the gauge multiplet.

3.4.2 $U(1)$ Invariant Action with Counterterms

As shown in the earlier chapter, in the warped superspace formalism, as far as the bulk action is concerned, we can always absorb the usual superpotential G into $H_i\partial_5\Phi^i$ by a redefinition (we will come back to this issue later). Therefore we take the following standard rigid action

$$S = \int d^5x \left\{ \int d^4\theta \mathrm{e}^{-2kz} K + \left[\int d^2\theta \mathrm{e}^{-3kz} H_i \partial_5 \Phi^i + h.c. \right] \right\} \ . \tag{3.90}$$

By definition, now the superspace version of the essential Killing vector is

$$X^i(\Phi^k) = -3ik\Omega^{ij}H_j \ . \tag{3.91}$$

The following second SUSY transformation preserves the rigid action:

$$\delta\Phi^i = \frac{1}{2}e^{kz}\bar{D}^2\left[\Omega^{ij}K_j(\theta\eta + \bar{\theta}\bar{\eta})\right] - 12k\Omega^{ij}H_j\theta\eta - k\eta_s^A D_A\Phi^i \ . \tag{3.92}$$

Now let us assume that this action is also invariant under the following global symmetry:

$$\delta_\lambda\Phi^i = \lambda T^i \ , \tag{3.93}$$

where λ is a real constant parameter. The superfield dependent function $T(\Phi^i)$ must be a Killing vector to preserve the rigid action. So it is also an isometry on the target space. Our goal is to gauge the possible group generated by a closed set of such T.

To gauge a global symmetry in superspace, we first lift its parameter λ to a chiral superfield:

$$\delta_\Lambda\Phi^i = \Lambda T^i \ . \tag{3.94}$$

Mathematically this means the isometry group on the target space has been complexified.

Each vector T can be written as a gradient of a Killing potential or tri-holomorphic potential:

$$K_{ij^*}\bar{T}^{j^*} = i\mathrm{D}_i^{(T)} \tag{3.95}$$

$$\Omega_{ij}T^j = -i\mathrm{P}_i^{(T)} \ . \tag{3.96}$$

Note $\mathrm{D}(\Phi^i, \bar{\Phi}^{j^*})$ and $\mathrm{P}(\Phi^i)$ now are functions of superfields too.

The action (3.90) is not invariant under the super gauge transformation (3.94) though. The Kähler potential in fact changes into

$$\delta K = \Lambda\Upsilon + \bar{\Lambda}\bar{\Upsilon} - i(\Lambda - \bar{\Lambda})\mathrm{D}^{(T)} \ , \tag{3.97}$$

where $\Upsilon = T^i K_i + i\mathrm{D}^{(T)}$ is a holomorphic function. Although the first two terms correspond to a Kähler transformation, thus preserve the action under $d\theta^4$ integration, the last term does not vanish. To compensate this change, we need a real counterterm $\Gamma_V(\Phi^i, \bar{\Phi}^{j^*}, V)$. It can be solved as the standard form as in the 4-d $\mathcal{N} = 1$ case:

$$\Gamma_V(\Phi^i, \bar{\Phi}^{j^*}, V) = \frac{e^{\frac{i}{2}V\mathcal{O}} - 1}{\frac{i}{2}V\mathcal{O}} V\mathrm{D}^{(T)} \ , \tag{3.98}$$

where

$$\mathcal{O} \equiv T^i\partial_i - \bar{T}^{j^*}\partial_{j^*} \ . \tag{3.99}$$

Furthermore, the rigid superpoential is not invariant under (3.94) either, so another chiral counterterm is needed.

To find it, we integrate the gauge variation by parts:

$$\begin{aligned}
\delta_\Lambda S &= \delta_\Lambda \int d^5x \mathrm{e}^{-3kz} \int d^2\theta (H_i\partial_5\Phi) + h.c. \\
&= \int d^5x \mathrm{e}^{-3kz} \int d^2\theta \left[H_{i,j}\delta_\Lambda\Phi^j\partial_5\Phi^i + H_i\partial_5\left(\delta_\Lambda\Phi^i\right)\right] + h.c. \\
&= \int d^5x \mathrm{e}^{-3kz} \int d^2\theta \left[\Lambda\Omega_{ij}T^j\partial_5\Phi^i + 3k\Lambda H_i T^i\right] \\
&\quad + \int d^5x \int d^2\theta\partial_5[\mathrm{e}^{-3kz}\Lambda H_i T^i] + h.c. \\
&= \int d^5x \mathrm{e}^{-3kz} \int d^2\theta \left[-i\Lambda\mathrm{P}_i^{(T)}\partial_5\Phi^i + 3k\Lambda H_i T^i\right] \\
&\quad + \int d^5x \int d^2\theta\partial_5[\mathrm{e}^{-3kz}\Lambda H_i T^i] + h.c.
\end{aligned}$$

$$
\begin{aligned}
\Longrightarrow \delta_\Lambda S &= \int d^5x \mathrm{e}^{-3kz} \int d^2\theta \left[-i\Lambda\partial_5 \mathrm{P}^{(T)} + 3k\Lambda H_i T^i\right] \\
&\quad + \int d^5x \int d^2\theta \partial_5 \left[\mathrm{e}^{-3kz}\Lambda H_i T^i\right] + h.c. \\
&= \int d^5x \mathrm{e}^{-3kz} \int d^2\theta \left[i\mathrm{P}^{(T)}\partial_5\Lambda + 3k\Lambda(H_i T^i - i\mathrm{P}^{(T)})\right] \\
&\quad + \int d^5x \int d^2\theta \partial_5 \left[\mathrm{e}^{-3kz}\Lambda(H_i T^i - i\mathrm{P}^{(T)})\right] + h.c. \; .
\end{aligned}
\tag{3.100}
$$

So the second counterterm should be

$$
\Gamma_\chi(\Phi^i, \chi) = -\chi \mathrm{P}^{(T)} \; . \tag{3.101}
$$

Adding it to the superpotential removes the $i\mathrm{P}^{(T)}\partial_5\Lambda$ term in (3.100). The variation of the combined action then becomes:

$$
\delta_\Lambda S = \int d^5x \Big\{ \mathrm{e}^{-3kz} 3k \int d^2\theta \Lambda(H_i T^i - i\mathrm{P}^{(T)}) + \int d^2\theta \partial_5 \Big[\int d^2\theta \Lambda (H_i T^i - i\mathrm{P}^{(T)}) \Big] + h.c. \Big\} \; . \tag{3.102}
$$

Therefore the action with counterterms Γ_V and Γ_χ is invariant provided the following condition is met:

$$
i\mathrm{P}^{(T)} = H_i T^i = \frac{i}{3k} \Omega_{ij} X^i T^j \; . \tag{3.103}
$$

It matches the component constraint (3.28). Thus only the tri-holomorphic isometries commuting with the essential Killing vector X can be gauged; and the corresponding holomophic potential has no unfixed constant.

In superspace formalism, (3.103) is the only constraint on the tri-holomorphic Killing vector T^i. There is no counterpart for the other component constraint (3.29). In Sec. 4.8, we will show that a surface term present in the superspace action but absent in components is responsible for this difference.

Adding both counterterms, the gauge invariant action become

$$S = \int d^5x \quad \left\{ \int d^4\theta \mathrm{e}^{-2kz}(K + \Gamma_V) + [\int d^2\theta \mathrm{e}^{-3kz}(H_i \partial_5 \Phi^i + \Gamma_\chi) + h.c.] \right\}$$
$$+ \int d^4\theta e^{-2kz} \frac{1}{g^2}(\chi + \bar{\chi} - \partial_5 V)^2 + \left[\int d^2\theta \frac{1}{4g^2} W^\alpha W_\alpha + h.c. \right] \Bigg\} . (3.104)$$

One can further add a Chern-Simons action to it.

$$S_{CS} = \int d^5x \left\{ \left[\int d^2\theta \chi W^\alpha W_\alpha + h.c. \right] - \frac{2}{3} \int d^4\theta \mathrm{e}^{-2kz} (\partial_5 V - \chi - \bar{\chi})^3 \right.$$
$$\left. - \frac{2}{3} \left[\int d^4\theta (\partial_5 V D_\alpha V W^\alpha - V D_\alpha \partial_5 V W^\alpha) + h.c. \right] \right\} . \quad (3.105)$$

This is the warped superspace generalization of [35]. The first and the third terms in (3.105) are topological, and will generate the bosonic Chern-Simons action in the Wess-Zumino gauge. The second term is gauge invariant and is required by the second supersymmetry. Since it manifestly carries the warp factor, this term is not topological.

Now we need to find the proper second SUSY transformations to preserve the combined action.

In $\mathcal{N} = 1$ superspace, hypermultiplet is only realized semi-off-shell. The commutator $[\delta_\epsilon, \delta_\eta]\chi^i$ closes only after using the Dirac equation, which changes when the gauge coupling is included. Therefore we must modify the second SUSY transformation on hypermultiplets:

$$\delta\Phi^i = \frac{1}{2} e^{kz} \bar{D}^2 \left[\Omega^{ij} \left(K_j + \frac{\partial \Gamma_V}{\partial \Phi^j} \right) (\theta\eta + \bar{\theta}\bar{\eta}) \right] - 12k\Omega^{ij} H_j \theta\eta - k\eta_s^A D_A \Phi^i \ . \quad (3.106)$$

The transformation on vector multiplet will just take the same form as in the free case, because SUSY algebra there is closed off-shell.

Under the modified SUSY transformations, the gauged action (3.104) can be shown as invariant, provided the constraint (3.103) holds.

3.4.3 Generation of Superpotential Revisited

In $\mathcal{N}=1$ superspace, the Fayet's mechanism can be formulated in a simple way. The relevant part of action is

$$\begin{aligned} S &= \int d^5x \left\{ \int d^4\theta \mathrm{e}^{-2kz} \left[\frac{1}{g^2}(\chi + \bar{\chi} - \partial_5 V)^2 + \Gamma_V \right] \right. \\ & \left. + \left[\int d^2\theta \mathrm{e}^{-3kz} \Gamma_\chi + h.c. \right] + ... \right\} \ . \end{aligned} \tag{3.107}$$

In the Wess-Zunimo gauge, we have

$$\Gamma_V = V\mathrm{D} + \frac{i}{4}V^2(T^i\partial_i - \bar{T}^{j^*}\partial_{j^*})\mathrm{D} \ , \tag{3.108}$$

therefore

$$\begin{aligned} S &= \int d^5x \left\{ \int d^4\theta \mathrm{e}^{-2kz} \left[\frac{1}{g^2}(\chi + \bar{\chi} - \partial_5 V)^2 + V\mathrm{D} \right] \right. \\ & \left. + \left[- \int d^2\theta \mathrm{e}^{-3kz} \chi \mathrm{P} + h.c. \right] + ... \right\} \ . \end{aligned} \tag{3.109}$$

We now consider the following constant shift on the superfields:

$$\chi \rightarrow \chi + \frac{g\sigma}{2} \tag{3.110}$$

$$\mathrm{D} \rightarrow \mathrm{D} + d = \mathrm{D} + \frac{4k}{g}\sigma \ . \tag{3.111}$$

Note the overall factor $2/g$ difference from the component result can be absorbed by rescaling:

$$V \rightarrow gV \tag{3.112}$$

$$\chi \rightarrow g\chi \tag{3.113}$$

$$T^i \rightarrow 2T^i \tag{3.114}$$

$$\mathrm{D}^{(T)} \rightarrow 2\mathrm{D}^{(T)} \tag{3.115}$$

$$\mathrm{P}^{(T)} \rightarrow 2\mathrm{P}^{(T)} \; . \tag{3.116}$$

Such a combined shift generates a superpotential along with a surface term:

$$\begin{aligned}
\delta_\sigma S &= \int d^5x \left\{ \int d^4\theta \mathrm{e}^{-2kz} \left[2(-\partial_5 V)\frac{\sigma}{g} + gVd \right] + \left[\int d^2\theta \mathrm{e}^{-3kz} \frac{-g\sigma}{2} \mathrm{P} + h.c. \right] \right\} \\
&= \int d^5x \left\{ \left[\int d^2\theta \mathrm{e}^{-3kz} \frac{-g\sigma}{2} \mathrm{P} + h.c. \right] \right. \\
&\quad + \int d^5x \partial_5 \left\{ \int d^4\theta \mathrm{e}^{-2kz} \left[-\frac{2\sigma}{g} V \right] \right\} \; .
\end{aligned} \tag{3.117}$$

So with the spontaneously generation of superpotential, a 4-d $\mathcal{N}=1$ Fayet-Iliopoulos term is induced on the boundary. Turning this around, the generation of the bulk superpotential is actually due to the change of the boundary F-I term!

We have to distinguish two facts here. The first observation is: to stabilize the vacuum, shifting χ needs a compensating shift on the Killing potential D. As a result, one can not break supersymmetry spontaneously by adding a bulk F-I term. This fact is due to the warping of AdS_5 and disappears in the flat case.

The second observation is: to preserve supersymmetry, shifting χ needs a tuning on the surface-localized F-I term. So one can still break both supersymmetries by adding a surface F-I term. This is purely due to the existence of boundary. This effect is still true when we set $k = 0$ while keeping the space-time boundary[2].

One may wonder why this boundary F-I term never showed up in the component formalism.

The reason is: the component action for the gauge sector differs from the superspace one by the following surface term:

$$\int d^5x \mathrm{e}^{-4kz} Y = \int d^4x e^{-4kz} (\Sigma D + \frac{1}{2}\lambda_1\lambda_2 + \frac{1}{2}\bar{\lambda}_1\bar{\lambda}_2) \ . \tag{3.118}$$

This is called Gibbons-Hawking-York term [36–38], or Y-term for short. The flat space version of this Y-term has been found by Belyaev [39] in the Mirabelli-Peskin model. The field dependent function D in the off-shell formalism is the auxiliary field of the vector multiplet. In the on-shell formalism, it should be replaced by

$$D = \partial_5\Sigma + 2k\Sigma - g\mathrm{D}^{(T)} \tag{3.119}$$

The Y-term should be added to the component action to match the superspace one. Since superspace action is $\mathcal{N} = 1$ manifestly invariant and the Y-term is not supersymmetric, we have to conclude that the previous component action for the gauge sector is only $\mathcal{N} = 1$ invariant in the bulk, to be fully invariant both in the bulk and on the boundary, Y-term is required.

Using the fact that D in (3.119) is invariant under the combined shift, it is easy

[2] As we will see in Ch. 4, the structure of the AdS boundary is different from the flat space case. But it does not change the story here.

to check that the Y-term (3.118) really generates the correct F-I term:

$$\delta_\sigma Y = \int d^4x e^{-4kz}\left(\frac{1}{2}\sigma D\right) \tag{3.120}$$

In the rest of this section, we provide an alternative way to understand Fayet's mechanism. Let us study the 2nd SUSY transformations before and after the shift (3.110):

$$\delta_\eta V \quad\rightarrow\quad \delta_\eta V + 2\sigma\theta\eta + 2\sigma\bar{\theta}\bar{\eta} \tag{3.121}$$

$$\delta_\eta \chi \quad\rightarrow\quad \delta_\eta \chi \tag{3.122}$$

$$\delta_\eta \Phi^i \quad\rightarrow\quad \delta_\eta \Phi^i \ . \tag{3.123}$$

We can use a compensating supergauge transformation with the following parameter to restore the 2nd SUSY in the vector sector:

$$\Lambda = -2\sigma\theta\eta \ . \tag{3.124}$$

Gauge invariance then requires transformations on hypermultiplets too:

$$\begin{aligned}
\delta_\eta \Phi^i \quad &\rightarrow\quad \delta_\eta \Phi + \Lambda T^i \\
&= \quad \frac{1}{2}e^{kz}\bar{D}^2\left[\Omega^{ij}\left(K_j + \partial_j \Gamma_V\right)\left(\theta\eta + \bar{\theta}\bar{\eta}\right)\right] \\
&\quad\ -\Omega^{ij}(12kH_j - 2\sigma \mathrm{P}_j)\theta\eta - k\eta_s^A D_A \Phi^i \theta\eta \ ,
\end{aligned} \tag{3.125}$$

which has exactly the same form as a new model with the shifted Killing vector.

Therefore we can understand the formula in Fayet's mechanism purely from a technique point of view. Shifting superfields changes the 2nd SUSY in the vector sector.

Restoring this supersymmetry transformation by a compensating gauge transformation will just generate the new SUSY transformation formulas[3] on the matter sector.

To conclude, for spontaneous superpotential generation, all results derived previously in the component formalism have been confirmed by the superspace method. Moreover, two new facts have been revealed. First, a superpotential in the bulk can be and should be viewed as arising from a boundary F-I term, this is also true for flat 5-d space with boundaries. The second fact is that we may view Fayet's mechanism purely as a technical trick. In superspace, this mechanism is equivalent to using a supergauge transformation to generate a new SUSY transformation for a new model.

3.4.4 Reduction to the Component Formalism

Superspace formalism is a condensed language to study the formal aspects of sigma model. Unfortunately, it contains un-physical auxiliary fields which must be integrated out before studying the physical aspects. In this section, we illustrate the procedure of such a reduction.

For the vector multiplet, we will stick with the Wess-Zunimo gauge, in which the first counterterm has the following form:

$$\Gamma_V = gV\mathrm{D}^{(T)} + \frac{1}{2}g^2V^2K_{ij^*}T^i\bar{T}^{j^*} \ . \tag{3.126}$$

[3]As far as the SUSY transformations are concerned, shift (3.111) plays no role. The understanding is univeral for cases with 8 supercharges. For instance, Xiong [40] has used the similar procedure to produce the proper 4-d 2nd SUSY transformation after a dimension reduction from flat 5-d.

The action involving the auxiliary fields is

$$
\begin{aligned}
S_{F,D} \quad = \quad & \int d^5x \mathrm{e}^{-4kz} \Big\{ 2k\Sigma D - \Sigma \partial_5 D + \frac{1}{2} D^2 + gD\mathrm{D}^{(T)} \\
& \qquad +2gF\bar{F} - 2F\mathrm{P}^{(T)} - 2g\bar{F}\bar{\mathrm{P}}^{(T)} + L_{\mathcal{F}} \quad \Big\} , \qquad (3.127)
\end{aligned}
$$

where we have already made the rescaling (3.112 - 3.116).

The auxiliary fields can be solved as

$$F = g\bar{\mathrm{P}}^{(T)} \qquad (3.128)$$

$$D = -\partial_5 \Sigma + 2k\Sigma - g\mathrm{D}^{(T)} \qquad (3.129)$$

$$\mathcal{F}^i = \frac{1}{2}\Gamma^i_{jk}\chi^j\chi^k + \Omega^i{}_{j^*}\left[-\partial_5 A^{*j^*} - i\bar{X}^{j^*} + ig\Sigma\bar{T}^{j^*} + gv_5\Sigma\bar{T}^{j^*}\right] . \qquad (3.130)$$

The SUSY transformations in superspace break the Wess-Zunimo gauge. To restore it, a compensating gauge transformation is necessary:

$$V \rightarrow V + i\Lambda - i\bar{\Lambda} \qquad (3.131)$$

$$\chi \rightarrow \chi + i\partial_5 \Lambda \qquad (3.132)$$

$$\Phi^i \rightarrow \Phi^i + \Lambda T^i , \qquad (3.133)$$

where the chiral superfield Λ has ϵ and field dependence.

For instance, to restore the Wess-Zunimo gauge after the first SUSY, we need

$$\Lambda_1 = -i\theta\sigma^a\bar{\epsilon}\delta^m_a v_m - \theta^2 \mathrm{e}^{-\frac{3}{2}kz}\bar{\epsilon}\bar{\lambda}_1 . \qquad (3.134)$$

To compensate the second SUSY (3.159), we need

$$
\begin{aligned}
\Lambda_2 \quad = \quad & 2i\theta\eta\Sigma + k\delta_{ma}\delta^n_b x^m \theta\sigma^b\bar{\sigma}^a v_n \\
& + \frac{1}{2}\mathrm{e}^{-\frac{kz}{2}}\theta^2\eta\lambda_2 + ik\mathrm{e}^{-\frac{3kz}{2}}\delta_{ma}x^m\theta^2(\eta\sigma^a\bar{\lambda}_1) . \qquad (3.135)
\end{aligned}
$$

The following modified SUSY transformations preserve the Wess-Zumino gauge:

$$\delta_\epsilon V = (\epsilon Q + \bar{\epsilon}\bar{Q}) \times V + i\Lambda_1 - i\bar{\Lambda}_1 \tag{3.136}$$

$$\delta_\epsilon \chi = (\epsilon Q + \bar{\epsilon}\bar{Q}) \times \chi + i\partial_5 \Lambda_1 \tag{3.137}$$

$$\delta_\epsilon \Phi^i = (\epsilon Q + \bar{\epsilon}\bar{Q}) \times \Phi^i + \Lambda_1 T^i \ , \tag{3.138}$$

$$\delta_\eta V = 2(\chi + \bar{\chi} - \partial_5 V)(\theta\eta + \bar{\theta}\bar{\eta}) - k\eta_s^A D_A V + i\Lambda_2 - i\bar{\Lambda}_2 \tag{3.139}$$

$$\delta_\eta \chi = -e^{2kz}\eta W - k\eta_s^A D_A \chi + i\partial_5 \Lambda_2 \tag{3.140}$$

$$\begin{aligned} \delta_\eta \Phi^i = {} & \frac{1}{2} e^{kz} \bar{D}^2 \left[\Omega^{ij} \left(K_j + \partial_j \Gamma_V\right) (\theta\eta + \bar{\theta}\bar{\eta})\right] \\ & -12k\Omega^{ij} H_j \theta\eta - k\eta_s^A D_A \Phi^i + \Lambda_2 T^i \ . \end{aligned} \tag{3.141}$$

After the θ expansion and plugging in (3.128, 3.129), on-shell SUSY transformations in 2-component spinor notation can be produced. The detailed expressions are too complicated so we ignore them here.

These tedious transformations can be rewritten in a concise form when we use 4-component spinor notation. The two SUSY transformation parameters ϵ and η are collected into Killing spinors $\epsilon_\pm$ as (2.19) and (2.20). Two chiral gauginos λ_1, λ_2 also form the following symplectic Majorana gauginos:

$$\lambda_+ = \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix} \ , \qquad \lambda_- = \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \ . \tag{3.142}$$

Then the SUSY transformations match (3.11 – 3.13) exactly.

In order to match the component action in the bulk, the Y-term is needed. We will return to this point in the Ch. 4.

3.4.5 Non-Abelian Case

To generalized $\mathcal{N}=1$ superspace method to the non-Abelian gauged case, we first promote superfields to Lie algebra valued quantities:

$$\begin{aligned} V &= V^{(a)}t^{(a)} && (3.143)\\ \chi &= \chi^{(a)}t^{(a)} && (3.144)\\ \Lambda &= \Lambda^{(a)}t^{(a)} \ , && (3.145)\end{aligned}$$

where the matrices ${t^{(a)i}}_j$ have the following commutation relations:

$$[t^{(a)}, t^{(b)}] = if^{abc}t^{(c)} \ . \tag{3.146}$$

The finite gauge transformation on the gauge multiplet looks like

$$\begin{aligned} V &\to \mathrm{e}^{-i\bar{\Lambda}}\mathrm{e}^{V}\mathrm{e}^{i\Lambda} && (3.147)\\ \chi &\to \mathrm{e}^{-i\Lambda}\chi\mathrm{e}^{i\Lambda} + \mathrm{e}^{-i\Lambda}\partial_5\mathrm{e}^{i\Lambda} \ . && (3.148)\end{aligned}$$

Gauge covariant field strengths are defined as

$$\begin{aligned} \mathcal{W}_\alpha &= \mathrm{W}_\alpha = \frac{-1}{4}\bar{D}^2(\mathrm{e}^{-V}D_\alpha\mathrm{e}^{V}) && (3.149)\\ \bar{\mathcal{W}}_{\dot\beta} &= \mathrm{e}^{-V}\bar{\mathrm{W}}_{\dot\beta}\mathrm{e}^{V} = \mathrm{e}^{-V}\left[\frac{1}{4}D^2(\mathrm{e}^{V}\bar{D}_{\dot\beta}\mathrm{e}^{-V})\right]\mathrm{e}^{V} \ . && (3.150)\end{aligned}$$

Note, straight W and $\bar{\mathrm{W}}$ are related by hermitian conjugation, while curly $\mathcal{W}$ and $\bar{\mathcal{W}}$ are not.

There is another covariant quantity, first noticed by Hebecker [41]:

$$Z = \mathrm{e}^{-V}\partial_5 \mathrm{e}^{V} - \mathrm{e}^{-V}\bar{\chi}\mathrm{e}^{V} - \chi \ . \tag{3.151}$$

Z is not Hermitian except in the Abelian case, but it is related to its conjugate:

$$\bar{Z} = \mathrm{e}^{V} Z \mathrm{e}^{-V} \ . \tag{3.152}$$

Under a finite gauge transformation, these quantities transform covariantly:

$$\mathcal{W}_\alpha \ \rightarrow \ \mathrm{e}^{-i\Lambda}\mathcal{W}_\alpha \mathrm{e}^{i\Lambda} \tag{3.153}$$

$$\bar{\mathcal{W}}_{\dot\beta} \ \rightarrow \ \mathrm{e}^{-i\Lambda}\bar{\mathcal{W}}_{\dot\beta} \mathrm{e}^{i\Lambda} \tag{3.154}$$

$$Z \ \rightarrow \ \mathrm{e}^{-i\Lambda} Z \mathrm{e}^{i\Lambda} \ . \tag{3.155}$$

The Bianchi identity

$$\mathscr{D}^\alpha \mathcal{W}_\alpha = \bar{\mathscr{D}}_{\dot\beta}\bar{\mathcal{W}}^{\dot\beta} \tag{3.156}$$

has the following explicit form:

$$D^\alpha \mathrm{W}_\alpha + (\mathrm{e}^{-V} D^\alpha \mathrm{e}^{V})\mathrm{W}_\alpha - \mathrm{W}^\alpha(\mathrm{e}^{-V} D_\alpha \mathrm{e}^{V}) = \bar{D}_{\dot\beta}(\mathrm{e}^{-V}\bar{\mathrm{W}}^{\dot\beta}\mathrm{e}^{V}) \ . \tag{3.157}$$

The non-Abelian gauge sector in warped superspace has the following gauge invariant action:

$$S_{YM} = \int d^5x \left\{ \mathrm{Tr}\left[\int d^2\theta \frac{1}{4g^2}\mathrm{W}^\alpha \mathrm{W}_\alpha + h.c.\right] + e^{-2kz}\mathrm{Tr}\left[\int d^4\theta \frac{1}{g^2} Z^2\right]\right\} \ . \tag{3.158}$$

The action is invariant under $\mathcal{N}=1$ manifestly, and can be shown invariant under the following 2nd SUSY transformations:

$$
\begin{aligned}
\delta_\eta \mathrm{e}^V &= 2(\bar{\chi}\mathrm{e}^V + \mathrm{e}^V\chi - \partial_5 \mathrm{e}^V)(\theta\eta + \bar{\theta}\bar{\eta}) - k\eta_S^A D_A \mathrm{e}^V && (3.159)\\
\delta_\eta \chi &= -\mathrm{e}^{2kz}\eta^\alpha \mathrm{W}_\alpha - k\eta_S^A D_A \chi \; . && (3.160)
\end{aligned}
$$

To couple the gauge sector to the sigma model, we study the following infinitesimal gauge transformation, linear in Λ:

$$
\begin{aligned}
\delta_\Lambda \mathrm{e}^V &= -i\bar{\Lambda}\mathrm{e}^V + i\mathrm{e}^V\Lambda && (3.161)\\
\delta_\Lambda \chi &= i\partial_5\Lambda - [i\Lambda, \chi] && (3.162)\\
\delta_\Lambda \Phi^i &= \Lambda^{(a)} T^{(a)i} \; , && (3.163)
\end{aligned}
$$

where the $T^{(a)}$ are target space Killing vectors generating global isometries. They are related to the Lie algebra generators $t^{(a)}$ in the following formula:

$$
T^{(a)} = -i\Phi^j \; t^{(a)k}{}_j \, \frac{\partial}{\partial \Phi^k} \; , \tag{3.164}
$$

so that

$$
[T^{(a)}, T^{(b)}] = -f^{abc} T^{(c)} \; . \tag{3.165}
$$

Like the Abelian case, the rigid sigma model action is not invariant when $\Lambda^{(a)}$ are chiral superfields. The two counterterms Γ_V and Γ_χ in this case are

$$
\Gamma_V = \frac{e^{\frac{i}{2}V^{(a)}\mathcal{O}^{(a)}} - 1}{\frac{i}{2}V^{(b)}\mathcal{O}^b} V^{(c)} \mathrm{D}^{(c)} \; , \tag{3.166}
$$

and

$$
\Gamma_\chi = -\chi^{(a)} \mathrm{P}^{(a)} \; , \tag{3.167}
$$

where

$$\mathcal{O}^{(a)} \equiv T^{(a)i}\frac{\partial}{\partial\Phi^i} - \bar{T}^{(a)j^*}\frac{\partial}{\partial\bar{\Phi}^{j^*}} \ . \tag{3.168}$$

The following combined action is invariant under (3.161) as long as the $T^{(a)}$ are triholomorphic and commute with the essential Killing vector $X^i = 3ik\Omega^{ij}H_j$:

$$S = S_{YM} + \int d^5x \left\{ \int d^4\theta \mathrm{e}^{-2kz}(K + \Gamma_V) + [\int d^2\theta \mathrm{e}^{-3kz}(H_i\partial_5\Phi^i + \Gamma_\chi) + h.c.] \right\} \ . \tag{3.169}$$

The action is also invariant under the 2nd SUSY transformations as (3.159), (3.160) and

$$\delta\Phi^i = \frac{1}{2}e^{kz}\bar{D}^2\left[\Omega^{ij}\left(K_j + \frac{\partial\Gamma_V}{\partial\Phi^j}\right)(\theta\eta + \bar{\theta}\bar{\eta})\right] - 12k\Omega^{ij}H_j\theta\eta - k\eta_s^A D_A\Phi^i \ . \tag{3.170}$$

3.5 Conclusions

In this chapter, we constructed the gauged supersymmetric nonlinear sigma model in AdS_5 using two methods: the component formalism and warped $\mathcal{N} = 1$ superspace. The component action only contains physical fields so it is more transparent as far as the dynamics of physical fields is concerned. $\mathcal{N} = 1$ superspace, on the other hand, seems to be more powerful to handle formal issues, taking the superpotential generation problem as an example. In component language, we find a technical solution to the puzzle: by shifting the gauge scalar Σ and the $U(1)$ Killing potential D, the potential can be stabilized and the SUSY transformations are unbroken. What superspace method finds, as shown in Sec. 3.4.3, is something completely missed in the component formalism: boundary effect. In superspace language, we can clearly see that shifting the gauge scalar Σ by a constant generates a surface F-I term as well! Both methods reach the same physical conclusion: for a given model, AdS_5 SUSY forbids any free F-I term. What is really interesting here is the new point of view provided by the superspace approach: Fayet's spontaneous superpotential generation corresponds to tuning F-I terms on the surface.

From both component and superspace approaches, we have derived the same constraint: only the tri-holomorphic isometries that preserve the essential Killing vector X of the rigid sigma model can be gauged. If we denote the generator of this restricted isometry group as T, then AdS SUSY requires the holomorphic potential $\mathrm{P}^{(T)}$ to be completely fixed. In another words, there is no unfixed integration constant even when T is an Abelian factor. This result differs from the flat case.

In this chapter, we have already revealed several problems related to the boundary effect. In the next chapter, boundary problems will be studied systematically.

Chapter 4

Boundary Conditions and the Boundary Induced Theory

4.1 Introduction to Boundary Problems

First, let us discuss a simple example living in a space-time region with boundaries: a single charged particle moving in 3+1 space-time filled with electromagnetic field. The following action describes the particle's motion between the start point $s_1(x_1, t_1)$ and the end point $s_2(x_2, t_2)$:

$$S = \int_{s_1}^{s_2} ds \left\{ -m + eA_\mu \frac{dx^\mu}{ds} \right\} \ , \tag{4.1}$$

where s is the length parameter along the world-line, x^μ are coordinates.

The Euler-Lagrange equation can be derived from variating the action:

$$m\frac{d^2x^\mu}{ds^2} - eF^{\mu\nu}\frac{dx_\mu}{ds} = 0 \ . \tag{4.2}$$

This equation is invariant under the following gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \lambda \ . \tag{4.3}$$

However under (4.3), the action changes :

$$S \to S + e\int_{s_1}^{s^2} ds\ \partial_\mu \lambda \frac{dx^\mu}{ds} = S + e\lambda(2) - e\lambda(1) \ . \tag{4.4}$$

When s_1 and s_2 are fixed end-points, the change of S is a constant shift. Classically, such a constant shift in the action means nothing. We use this simple example to illustrate the following question: what is the consistent boundary condition?

Our philosophy is that both the equation of motion and the boundary condition should be derived from the variational principle.

The variation of the action (4.1) generates a boundary term:

$$\begin{aligned}\Delta S &= \int_{s_1}^{s^2} ds(\mathrm{EOM})\cdot \Delta x^\mu + d(A_\mu \Delta x^\mu) \\ &= \int_{s_1}^{s^2} ds(\mathrm{EOM})\cdot \Delta x^\mu + A_\mu \Delta x^\mu|_{s_2} - A_\mu \Delta x^\mu|_{s_1} \ .\end{aligned} \tag{4.5}$$

To produce the equation of motion without tricky boundary localized terms, the following boundary condition must be taken:

$$(A_\mu \Delta x^\mu)|_{s=s_1,s_2} = 0 \ . \tag{4.6}$$

The usual textbook approach is restricted to fixed end-point variations, so that

$$\Delta x^\mu|_{s_1,s=2} = 0 \ . \tag{4.7}$$

On the other hand, when Δx^μ are not fixed at end-points, for instance when we discuss a class of theories with their world-line end-points varying on a space-time boundary, the general boundary condition (4.6) is necessary.

We can require the gauge invariance of the boundary condition. Then the gauge variation of (4.6) produces a constraint

$$\partial_\mu \lambda \Delta x^\mu = 0 \ . \tag{4.8}$$

For general Δx^μ, this restricts $\partial_\mu \lambda(s_1) = \partial_\mu \lambda(s_2) = 0$. As a result, gauge freedom on the boundary is lost.

We should point out that the result can not be trusted if the boundary is an artificial one. In that case the theory should be embedded into a larger closed system and the contribution from "outside" may compensate what it lost on the "boundary".

This example reveals some general questions for field theories living in a space-time region with boundaries. What is the boundary condition required for the consistency of variational principle? What is the self-consistency constraint on the boundary condition? And the last but not least: does the boundary break symmetries in the bulk?

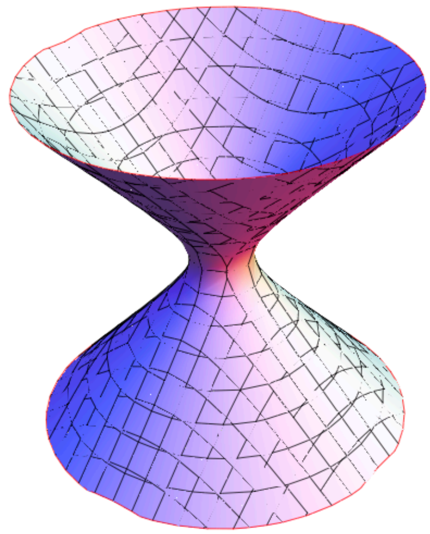

Figure 4.1: AdS_2: The top and bottom circles are time-like and correspond to the boundary at infinity. In higher dimensional AdS_n, these two circles are connected.

4.2 Boundary of AdS$_5$

5-d anti-de Sitter space has a boundary at infinity. This boundary can be best understood using a technique called conformal mapping. Here we give a very brief review. For a more detailed discussion, we refer readers to the classic book by Hawking and Ellis [42].

To fully describe AdS_5, one must use global coordinates. We may solve the embedding equation (2.2) as:

$$Y_0 = R \sec\theta \cos\tau \tag{4.9}$$

$$Y_i = R \tan\theta \Omega_i \tag{4.10}$$

$$Y_6 = R \sec\theta \sin\tau \ , \tag{4.11}$$

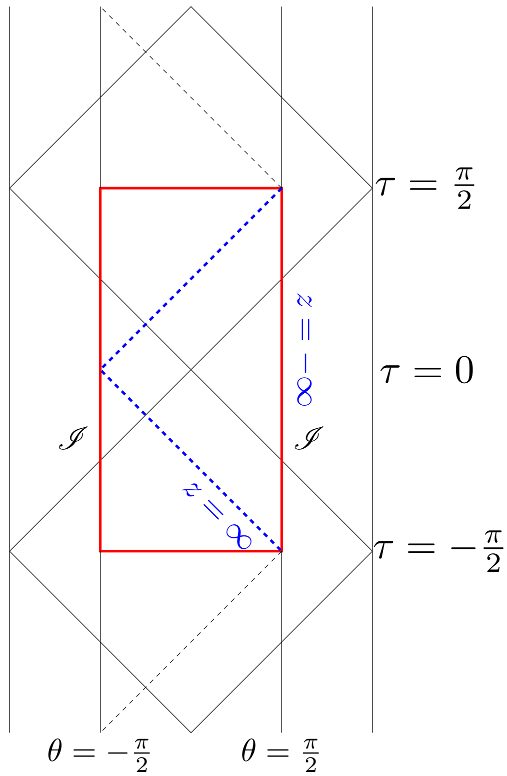

Figure 4.2: $CAdS_2$ Embedded in ESU: the AdS_2 is the red region. The dashed line show the separation of Poincare patches. Vertical lines $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$ are boundaries of AdS_2. Only half of which can be covered by a single Poincare patch.

where $i = 1, ..5$ and $\sum_i (\Omega_i)^2 = 1$. Coordinates here take the range $0 \leq \tau < 2\pi$ and $0 \leq \theta < \frac{\pi}{2}$. The AdS_5 metric in global coordinates becomes

$$ds^2 = \frac{1}{k^2 \cos^2 \theta}(-d\tau^2 + d\theta^2 + \sin^2 \theta d\Omega^2) \ . \tag{4.12}$$

By multiplying a conformal factor $k^2 \cos^2 \theta$, this metric can be mapped into half region of the Einstein Static Universe (ESU), whose topology is $R \times S^{n-1}$. Because a conformal mapping preserves the conformal structure, any conformal invariant field theory in AdS_n can be embedded in the ESU, with proper boundary conditions specified on the AdS_n boundary $\mathscr{I}$ [43].

Boundary conditions (asymptotic conditions) for individual bulk fields have been further studied in global coordinates [44–47]. These results are entries of the "dictionary" on AdS/CFT correspondence [48]. For detailed review on this topic, we refer readers to [49].

Although global coordinates and conformal mapping explain the meaning of AdS_n boundary beautifully, they are not satisfying for our practical use. The essential reason is that global coordinates do not foliate the AdS_5 in a 4-d Lorentz covariant manner. Promoting this coordinate system to $\mathcal{N} = 1$ superspace is not simple.

We will continue to use horospherical coordinates and the metric as (2.1). However, this coordinate system only covers half of AdS_5 space, called the Poincare patch. As a result, as shown in figure 4.2, only half of space-time boundary is covered. More problematic, it instead creates an artificial boundary at $z = +\infty$. Fortunately, as pointed out by Hawking and Ellis [42], it is in fact adequate to discuss bulk field theory within just one Poincare patch. Also, since AdS_5 can be covered by two Poincare patches, the artificial boundary at $z = +\infty$ in the first patch can be canceled by the second one's. Inspired by [50], we will first take a Poincare patch and truncated it by two regulators: 4-d flat branes locate at $z = z_-$ and $z = z_+$. Then after the derivation of boundary conditions in general case, we can take $z_- \to -\infty$ limit to recover the true space-time boundary $\mathscr{I}$.

4.3 Hyper-Kähler Geometry on the Darboux Patch

Complex geometry on hyper-Kähler manifolds will play an essential role in 5-d hypermultiplets' boundary problems. In this section, we will discuss some mathematical aspects and derive some results for physics study later.

Let us start with the anti-symmetric tensor Ω_{ij}. It appears in the sigma model action in superspace (for comparison, it is invisible in the component action and only shows up in the on-shell SUSY transformations):

$$\Omega_{ij} = H_{j,i} - H_{i,j} = \nabla_i H_j - \nabla_j H_i \ . \tag{4.13}$$

From Ω_{ij}, a holomorphic symplectic 2-form can be defined:

$$\omega = \frac{1}{2}\Omega_{ij} dA^i \wedge dA^j \ . \tag{4.14}$$

Due to the covariant constant property of Ω_{ij}, ω is closed:

$$d\omega = \frac{1}{2}(\Gamma^p_{ki}\Omega_{pj} + \Gamma^p_{kj}\Omega_{ip}) dA^k \wedge dA^i \wedge dA^j = 0 \ . \tag{4.15}$$

As a result, we can apply the complex version of Darboux theorem, to write this anti-symmetric tensor in a standard constant form:

$$\Omega_{ij} = \begin{pmatrix} \Omega_{IJ} & \Omega_{I\hat{J}} \\ \Omega_{\hat{I}J} & \Omega_{\hat{I}\hat{J}} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \ . \tag{4.16}$$

This is called the **canonical form** of the symplectic structure Ω_{ij}. Note this expression is not just valid at a single point p. It in fact holds within the neighborhood U_p around p, called the Darboux patch. For simplicity, from now on in this thesis when we talk about Darboux patch we mean such a patch with specified Darboux coordinates. Since Ω_{ij} is constant on a Darboux patch, the following result is obvious:

$$\partial_k \Omega_{ij} = 0 \ . \tag{4.17}$$

To see the importance of (4.17), let us compare it to gravity. In a curved space, although at any point p we can find a local coordinate system so that $\Gamma^{\rho}_{\mu\nu} = 0$, the derivatives of $\Gamma^{\rho}_{\mu\nu}$, a.k.a. curvatures can not vanish. Mathematically, Darboux theorem reveals that the symplectic structure is secretly global. In physics, the relation (4.17) will dramatically simplify our analysis.

On a hyper-Kähler manifold, for each complex structure J to be metric compatible, Riemann curvatures are related by $\Omega^{i}{}_{j^*}$ as in (1.19). This is called **the compatibility condition**. To preserve each J along a target space isometry generated by T^i, the tensor $\nabla_l T^k$ will also be related by $\Omega^{i}{}_{j^*}$ as in (3.17). This is called **the tri-holomorphic Killing condition**. One can draw an analogy and require a diffeomorphism to preserve only Ω_{ij} but not necessarily g_{ij^*} and $\Omega^{i}{}_{j^*}$. For future discussion, we call such a requirement **the symplectic condition**:

$$\begin{aligned}\delta_\xi \Omega_{ij} &= -\nabla_i \xi^k \Omega_{kj} - \nabla_j \xi^k \Omega_{ik} \\ &\quad + \Gamma^k_{ip} \xi^p \Omega_{kj} + \Gamma^k_{jp} \xi^p \Omega_{ik} - \Gamma^p_{ki} \xi^k \Omega_{pj} - \Gamma^p_{kj} \xi^k \Omega_{ip} = 0 \ .\end{aligned} \tag{4.18}$$

On a Darboux patch, Christoffel symbols are also related by Ω_{ij}. This can be seen by taking the difference between $\nabla_k \Omega_{ij} = 0$ and $\partial_k \Omega_{ij} = 0$:

$$\Omega^{ip} \Gamma^q_{pk} \Omega_{jq} = \Gamma^i_{jk} \ . \tag{4.19}$$

In this thesis, we call this formula **the Darboux compatible relation**.

Relation (4.19) simplifies calculation a lot. For instance, the symplectic condition (4.18) becomes

$$-\nabla_i \xi^k \Omega_{kj} - \nabla_j \xi^k \Omega_{ik} = 0 \ . \tag{4.20}$$

We don't know whether Darboux patches can be extended consistently to cover the whole hyper-Kähler manifold. In principle this consistency should be discussed, because usually sigma models do have globally defined target spaces. However for boundary problems, we always look for some local reduction on the manifold, so a single Darboux patch will be applicable.

4.4 Boundary Conditions for Flat 5-d Hypermultiplets

Both the flat and warped 5-d supersymmetric sigma models require the target space to be hyper-Kähler manifolds. In this section, we will use $\mathcal{N} = 1$ superspace to study the flat 5-d rigid sigma model's boundary problem. Even in this flat case, by taking this brand new approach we will find some interesting results.

4.4.1 Boundary Conditions from the Variational Principle

To start, we write down the action in a flat bulk truncated by two co-dimension 1 flat branes located at $z = z_-$ and $z = z_+$:

$$S = \int d^4x \int_{z_-}^{z_+} dz \Big\{ (K)_{\theta^4} + \Big\{ (H_i \partial_5 \Phi^i + G)_{\theta^2} + h.c. \Big\} \Big\} \ . \tag{4.21}$$

Taking a field variation on the action produces

$$\begin{aligned} \Delta S &= \int d^4x dz \left\{ \Delta H_i \partial_5 \Phi^i + H_i \partial_5 \Delta \Phi^i \right\} |_{\theta^2} + ... \\ &= \int d^4x dz (\text{Bulk EOM}) \Delta \Phi^i \\ &\quad + \left\{ \int d^4x \left(H_i \Delta \Phi^i \right)_{\theta^2} + h.c. \right\} \Big|_{z_-}^{z_+} \ . \end{aligned} \tag{4.22}$$

Without imposing any boundary condition, the surface term gives the following component expansion:

$$S.T. = \int [\delta(z - z_+) - \delta(z - z_-)] H_{ij} \mathcal{F}^j \Delta A^i + ... \tag{4.23}$$

After integrating out the auxiliary $\mathcal{F}$, a strange $(\delta_z)^2$ term will show up in the component action. Although such term may be meaningful when boundary-localized external sources exist, we consider it un-physical in our uncoupled theory.

Thus without taking any further assumption, we take the natural boundary condition to be:

$$(H_i \Delta \Phi^i)|_{z=z_-,z_+} = 0 \ . \tag{4.24}$$

4.4.2 Solving Boundary Conditions

To solve (4.24), we may try the most naive Dirichlet boundary condition

$$\Phi^i| = c^i, \text{ for all } i \ . \tag{4.25}$$

However, this is too strong! We can take the flat hyper-Kähler example and expand out all superfields. The component boundary conditions then become:

$$\begin{aligned} A^i| &= c^i & (4.26) \\ \partial_5 A^i| &= \Omega^i{}_{j^*}\mathcal{F}^{j^*}| = 0 \ . & (4.27) \end{aligned}$$

According to an ordinary differential equation result, when both the value and derivative are given on the boundary, the system is over-constrained. This observation shows that we can at most have n sets of superfield boundary conditions instead of $2n$ as we naively guess.

To have these boundary conditions explicitly written down, we work in a Darboux patch. One plausible choice turns out to be

$$\begin{aligned} \Phi^{\hat{I}}| &= c^{\hat{I}} & (4.28) \\ H_I| &= 0 \ . & (4.29) \end{aligned}$$

In complex geometry, (4.28) defines a global reduction. The result is a sub-manifold with an inherited Kähler potential, so it is Kählerian. Strictly speaking, (4.29) should not be viewed as boundary conditions but rather consistency constraints on (4.28).

On a Darboux patch, proper solutions to holomorphic functions H_i do exist. For instance

$$H_I = \frac{1}{2}\Omega_{\hat{J}I}(\Phi^{\hat{J}} - c^{\hat{J}}) \tag{4.30}$$

$$H_{\hat{J}} = \frac{1}{2}\Omega_{K\hat{J}}\Phi^K \ . \tag{4.31}$$

Using the property $\partial_k\Omega_{ij} = 0$, we can check

$$\Phi^{\hat{I}}| = c^{\hat{I}} \Longrightarrow H_I| = 0 \ ; \tag{4.32}$$

therefore (4.29) is consistent with (4.28).

Why we can require physical functions H_i to archive specific boundary values we chose? The reason is as follows: EOMs in the bulk only depends on Ω_{ij}, not on H_i; a given Ω_{ij} can be produced from different H_i, hence (4.29) defines a type of preferred choices of H_i. We must note that even with (4.29), H_i are not unique. From now on, we should not take any special form of H_i, only consider (4.29) in general.

We should distinguish redefinitions on H_i from field redefinitions on complex scalars A^i, which correspond to target space coordinate transformations. Darboux coordinates, for instance, are archived by such field redefinitions. On the other hand, $H_i dA^i$ should be viewed as the connection of an Abelian principle bundle on the target space. After fixing the coordinate system, we still have a freedom to redefine H_i.

We can draw an analogy to the gauge theory. Ω_{ij} is like the invariant field strength tensor $F_{\mu\nu}$; and H_i plays the role of the gauge field V_μ. Hence (4.29) can be viewed as a gauge fixing condition. From now on, we use "axial gauge" to refer to (4.29).

It is important to note the difference though. In the gauge theory, we only have a freedom to gauge away one component of V_μ. However, within a Darboux patch, Ω_{ij} is actually "flat", and half of H's components are secretly in the "pure gauge", that is why we can "gauge" them away.

Let us investigate such redefinitions carefully. Adding a boundary term to the action (4.21) does not change EOMs in the bulk, therefore generates an equivalent action:

$$S' = S + \left[\int d^4x (P)_{\theta^2} + h.c. \right] \Bigg|_{z_-}^{z_+} . \tag{4.33}$$

By integrating S' by parts, we obtain

$$S' = \int d^4x \int_{z_-}^{z_+} dz \Big\{ (K)_{\theta^4} + \Big\{ \big[(H_i + \partial_i P)\, \partial_5 \Phi^i + G \big]_{\theta^2} + h.c. \Big\} \Big\} . \tag{4.34}$$

This clearly shows that a redefinition on H_i is induced. Thus the boundary conditions (4.29) can be archived by choosing the boundary term P, so that

$$H'_I| = H_I| - \partial_I P| = 0 . \tag{4.35}$$

As a conclusion, we found a local Darboux patch around any point $p : \ (\Phi^i = c^i)$ on the hyper-Kähler manifold. In this patch, the geometry reduction $\Phi^{\hat{I}}| = c^{\hat{I}}$ determines a set of $\mathcal{N} = 1$ boundary conditions of the flat 5-d non-linear sigma model. The boundary values of the hypermultiplets form n $\mathcal{N} = 1$ chiral multiplets in 4-d.

4.5 Warped 5-d Hypermultiplet

4.5.1 Boundary Conditions from Field Variations

We again set up our sigma model in a slice of warped space truncated by two 4-d flat branes at $z = z_- andz = z_+$.

The bulk action with a general superpotential G is

$$S = \int d^4x \int_{z_-}^{z_+} dz \Big\{ e^{-2kz}(K)_{\theta^4} + e^{-3kz}\Big\{ \big[H_i \partial_5 \Phi^i + G\big]_{\theta^2} + h.c. \Big\}\Big\} , \tag{4.36}$$

whose field variation is

$$\begin{aligned} \Delta S &= \int d^5x e^{-3kz} \left\{ \left[H_{i,j} \Delta\Phi^j \partial_5 \Phi^i + H_i \partial_5(\Delta\Phi^i) + G_i \Delta\Phi^i \right]_{\theta^2} + h.c \right] + ... \\ &= \int d^5x e^{-3kz} \left\{ \left[(H_{i,j} - H_{j,i})\, \Delta\Phi^j \partial_5 \Phi^i + (G_i + 3kH_i)\, \Delta\Phi^i \right]_{\theta^2} + h.c \right] + ... \\ &\quad + \int d^5x \left\{ e^{-3kz} (H_i \Delta\Phi^i)_{\theta^2} [\delta(z - z_+) - \delta(z - z_-)] + h.c. \right\} . \end{aligned} \tag{4.37}$$

Therefore the consistency of the variation principle requires the following boundary condition:

$$(H_i \Delta\Phi^i)|_{z=z_+,z_-} = 0 . \tag{4.38}$$

Following the flat space result, we can solve (4.38) on a Darboux patch U_p around $p : (\Phi^i = c^i)$ as:

$$\Phi^{\hat{I}}| = c^{\hat{I}} \tag{4.39}$$

$$H_I| = 0 . \tag{4.40}$$

This is the proper boundary condition set we have been looking for.

4.5.2 Equivalent Superspace Action

When we add a boundary term to the action (4.36), an equivalent action is generated as follows:

$$\begin{aligned} S' &= S + \int d^4x \Big[\mathrm{e}^{-3kz} (P)_{\theta^2} + h.c. \Big] \Big|_{z_-}^{z^+} \\ &= \int d^4x \int_{z_-}^{z_+} dz \Big\{ e^{-2kz} (K)_{\theta^4} + e^{-3kz} \Big\{ \big[(H_i + \partial_i P)\, \partial_5 \Phi^i + (G - 3kP) \big]_{\theta^2} + h.c. \Big\} \Big\} . \end{aligned} \tag{4.41}$$

Thus H_i transforms as a vector potential under the induced field redefinition, while the superpotential G transforms as a Goldstone. Two invariant physical quantities are:

$$\Omega_{ij} = H_{j,i} - H_{i,j} \tag{4.13}$$

$$X^i = i\Omega^{ij}(3kH_j + G_j) \ . \tag{4.42}$$

For any given choice of H_i that satisfies (4.13), the in-homogeneous tri-holomorphic Killing condition of X^i serves as the integrability condition for G. This allows us to pick any gauge. There are two useful choices:

- Axial gauge:

$$H_I| = 0 \ , \tag{4.40}$$

which is necessary to have $\Phi^{\hat{I}} = c^{\hat{I}}$ as consistency boundary conditions;

- Unitary gauge:

$$X^i = 3ik\Omega^{ij} H_j \ , \quad G = 0 \ , \tag{4.43}$$

which is useful to simplify calculations.

In general, a unitary gauge choice is not axial. Because combining (4.40) and (4.43) produces a constraint on the physical quantity X^i:

$$X^{\hat{I}}| = 0 \ . \tag{4.44}$$

However, based on a geometric observation, an unitary axial gauge is still possible if we locally rotate the Darboux coordinate frame[1], so that n coordinate directions $\partial_{\hat{I}} = \partial/\partial A^{\hat{I}}$ are orthogonal to the Killing vector X, therefore X^i is parallel to the sub-manifold $\mathcal{S} : \{A^{\hat{I}} = c^{\hat{I}}\}$. Examples already appear in non-supersymmetric sigma models. For instance, when $O(2)$ sigma model undergoes a spontaneous symmetry breaking, the π field is defined as the fluctuation along the angular θ-coordinate, which is generated by the rotational Killing vector. This Killing vector is parallel to the sub-manifold $\mathcal{S} : \ \{\rho = v\}$; and the σ field describing fluctuation along the orthogonal radial ρ-direction, is analogous to $A^{\hat{I}}$.

Although boundary conditions (4.39) and (4.40) are archived by adding boundary term P, once we find the proper $H'_i = H_i + \partial_i P$ so that $H'_I| = 0$, we can use H'_i to construct the bulk action, ignoring H_i totally; the role of P is then invisible. However, realizing a boundary condition by adding a boundary term is a general idea. For instance, we can use different term to archive other conditions. Further discussion in include in Sec. 4.9. For now we stick with (4.39) and (4.40) for simplicity.

[1] In mathematics books, the term "Darboux frame" usually refers to the moving frame constructed on a curve. To clarify, We here use the word "frame" to refer to $2n$ local coordinate directions on a Darboux patch.

4.6 Boundary Condition for Vector Multiplet

4.6.1 Vector Multiplet in Flat 5-d

We can study boundary conditions for free gauge theory too. For simplicity, here we only discuss Abelian case. The relevant part under a field variation is

$$\begin{aligned}
\Delta S &= -2\int d^5x \int d^4\theta(\chi+\bar{\chi}-\partial_5 V)\Delta\partial_5 V + ... \\
&= \int d^5x \int d^4\theta(\mathrm{EOM})\Delta V + \int d^4x \int d^4\theta(\chi+\bar{\chi}-\partial_5 V)\Delta V\Big|_{z_-}^{z_+} \ .
\end{aligned} \tag{4.45}$$

The $\mathcal{N}=1$ invariant boundary condition is then

$$(\chi| + \bar{\chi}| - \partial_5 V|)\Delta V| = 0 \ . \tag{4.46}$$

If we work in the Wess-Zunimo gauge, there are two solutions to this condition:

$$V| = 0 \ , \tag{4.47}$$

or

$$\chi| + \bar{\chi}| - \partial_5 V| = 0 \ . \tag{4.48}$$

- The choice (4.47) means the boundary value of 4-d $\mathcal{N}=1$ vector multiplet is 0.

Then on the boundary we only have a chiral multiplet χ left. Imposing the equation of motion of the auxiliary field

$$D| = \partial_5\Sigma| \ , \tag{4.49}$$

(4.47) becomes the following component conditions

$$v_m| = 0 \tag{4.50}$$

$$\lambda_1| = 0 \tag{4.51}$$

$$\partial_5 \Sigma| = 0 \tag{4.52}$$

$$\partial_5 \lambda_2| = 0 \tag{4.53}$$

$$\partial_m \partial_5 v_5| = 0 \ . \tag{4.54}$$

The last condition looks strange and needs an interpretation.

First, we notice this is a perfect example about gauge symmetry breaking by boundary conditions. To preserve (4.50), the 5-d gauge transformation parameter $f(x,z)$ must satisfy

$$\partial_m f(x,z) = 0 \ . \tag{4.55}$$

So f can only be a function on z; and there is still a residual gauge redundancy on v_5:

$$v_5 \to v_5 + \partial_5 f(z) \ . \tag{4.56}$$

However, since the gauge symmetry has already been broken, nothing stops us to get rid of the residual redundancy. So let us solve (4.54) as

$$\partial_5 v_5| = \text{const} \ . \tag{4.57}$$

Now the component set of boundary condition has a nice form of mixed Dirichlet-Neumann type.

- The second choice (4.48), on the other hand, produces component conditions as follows:

$$\Sigma| = 0 \tag{4.58}$$

$$\lambda_2| = 0 \tag{4.59}$$

$$F_{5m}| = 0 \tag{4.60}$$

$$\partial_5\lambda_1 = 0 \tag{4.61}$$

$$\partial_5 D| = 0 \ . \tag{4.62}$$

This set of boundary conditions is invariant under 5-d gauge transformation.

If we use the equation of motion of D, (4.62) becomes

$$\partial_5\partial_5\Sigma| = 0 \ . \tag{4.63}$$

Note (4.63) does not over-restrict Σ, actually it is just a consistency constraint on (4.58). When (4.58) holds, its derivative gives $\partial_m\Sigma| = 0$, thus the equation of motion of Σ on the boundary becomes

$$\partial_5\partial_5\Sigma| = -\partial^m\partial_m\Sigma| = 0 \ , \tag{4.64}$$

which is exactly (4.63). Since it is just a consistent secondary constraint, we can set it aside and the rest boundary condition set again becomes a mixed Dirichlet-Neumann type.

To conclude, for free $U(1)$ gauge theory living in flat 5-d, there are two possible boundary conditions. One breaks gauge invariance and the other preserves it. Both choices produce boundary conditions as mixed Dirichlet-Neumann type. This result can be generalized to non-Abelian gauge case.

4.6.2 AdS$_5$ Vector Multiplet

As in flat space, in warped $\mathcal{N}=1$ superspace, free $U(1)$ gauge theory requires the following $\mathcal{N}=1$ boundary condition:

$$(\chi| + \bar{\chi}| - \partial_5 V|)\Delta V| = 0 \ . \tag{4.65}$$

Same two solutions are obvious:

$$V| = 0 \ , \tag{4.66}$$

or

$$\chi| + \bar{\chi}| - \partial_5 V| = 0 \ . \tag{4.67}$$

The only differences show up in their component expansions.

- After integrating out the auxiliary field, the first choice corresponds to the following component conditions:

$$v_m| = 0 \tag{4.68}$$

$$\lambda_1| = 0 \tag{4.69}$$

$$\partial_5 \Sigma| - 2k\Sigma| = 0 \ . \tag{4.70}$$

To restore boundary conditions in either Dirichlet or Neumann form, one can redefine the gauge scalar field as

$$\tilde{\Sigma} \equiv \mathrm{e}^{-2kz}\Sigma \ , \tag{4.71}$$

then the last on-shell boundary condition becomes

$$\partial_5 \tilde{\Sigma}| = 0 \ . \tag{4.72}$$

Consistency of this condition under $\mathcal{N} = 1$ SUSY requires

$$\begin{aligned} \partial_5(\tilde{\lambda}_2) &\equiv \partial_5(\mathrm{e}^{-\frac{1}{2}kz}\lambda_2)| = 0 && (4.73) \\ \partial_5 v_5| &= \mathrm{const} \ , && (4.74) \end{aligned}$$

where we have discard the redundancy in v_5 since these boundary conditions already breaks gauge symmetry.

- As for the second choice (4.67) that preserves the gauge symmetry, after integrating out D, component boundary conditions are as follows:

$$\begin{aligned} \tilde{\Sigma}| &= 0 && (4.75) \\ \tilde{\lambda}_2 &= 0 && (4.76) \\ F_{m5}| &= 0 && (4.77) \\ \partial_5 \tilde{\lambda}_1 &\equiv \partial_5(\mathrm{e}^{-\frac{3}{2}kz}\lambda_1)| = 0 && (4.78) \\ \partial_5\partial_5\tilde{\Sigma} &= 0 \ , && (4.79) \end{aligned}$$

where (4.79) is the consistency constraint on (4.75).

To conclude, in AdS_5, we find two choices of boundary conditions as Dirichlet-Neumann mixed type, after proper field redefinitions.

4.7 Consistency of Boundary Conditions

Previously, for both hypermultiplets and vector multiplets living in a 5-d bulk truncated by 2 flat branes, we derived boundary conditions on $\mathcal{N} = 1$ superfields. These conditions passed consistency checks. However, superfields contain non-physical auxiliary fields; so a non-trivial question is: are these boundary conditions still consistent after we integrate out auxiliary fields? In the last section, we have seen vector theory passes this check and component fields are always in mixed Dirichlet-Neumann conditions. In this section, we will study hypermultiplets.

Immediately, we encounter something strange. By expanding the first condition in (4.38) and integrating out $\mathcal{F}$, we have

$$c^{\hat{I}} = A^{\hat{I}}| \tag{4.80}$$

$$0 = \chi^{\hat{I}}| \tag{4.81}$$

$$0 = \left[\Omega^{\hat{I}J} g_{J\hat{P}^*}(\partial_5 A^{*\hat{P}^*} + i\bar{X}^{\hat{P}^*}) + \Omega^{\hat{I}J} g_{JQ^*}(\partial_5 A^{*Q^*} + i\bar{X}^{Q^*}) + \frac{1}{2}\Gamma^{\hat{I}}_{JK}\chi^J\chi^K\right]\Bigg|. \tag{4.82}$$

The first two are Dirichlet conditions, while the last is a consistency condition on $\mathcal{N} = 1$ SUSY invariance. However, (4.82) can be improper, since it can also be viewed as a Neumann condition involving $\partial_5 A^{\hat{I}}$ hence over-restricts the system.

4.7.1 Over-determination of Boundary Conditions

We have to carefully investigate whether fixing both A and $\partial_5 A$ on the boundary over-determines the system. The answer is completely dependent on the structure of boundary.

It is not a problem for the flat space case. The reason is simple: in our flat space setup, there are two DISCONNECTED boundaries at $z = z_-$ and $z = z_+$. Specifying both A and $\partial_5 A$ on one of them is just the standard Cauchy problem, where the fifth dimension z plays the role of "time". As long as we keep the information on the second brane unspecified, Cauchy data can consistently "propagate" through the bulk to it.

A concrete example with 2 complex scalars is as follows: on the brane $z = z_-$, fix A^1 and $\partial_5 A^1$ along with $H_2 = 0$; on brane $z = z_+$, fix A^2, $\partial_5 A^2$ and $H_1 = 0$.

It is easy to check this condition satisfies our requirement $H_i \Delta\Phi^i| = 0$.

The same situation appears when we use two branes to truncate the warped space. In fact, in 2-brane scenario, one can always choose the following $\mathcal{N} = 1$ Cauchy-Cauchy boundary condition:

$$\boxed{\text{Fix } \Phi^{\hat{I}} \text{ on } z = z_- \text{ , while fixing } \Phi^{J} \text{ on } z = z_+} \tag{4.83}$$

Now let us send $z_\pm \to \pm\infty$ and discuss the true space-time boundary of AdS_5. As we claimed earlier, our Poincare patch only covers half of space with half its boundary; it must be accompanied by another copy. When we work in the covering

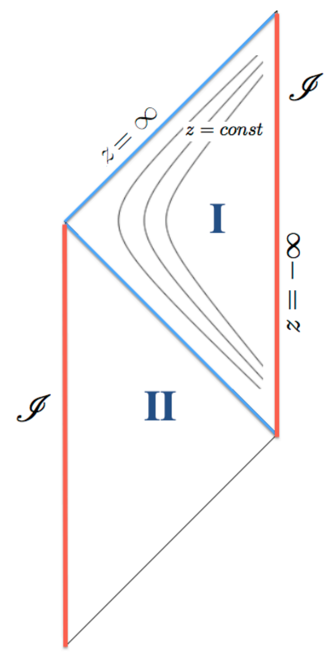

Figure 4.3: Two Poincare Patches: $z=-\infty$ is real space-time boundary while $z=\infty$ is artificial and will be canceled by another Poincare patch. In AdS_n with $n\geq 3$, boundary $\mathscr{I}$ is connected.

space $CAdS_5$ of AdS_5, the artificial boundary $z=z_+$ is canceled out while two real boundaries $z=z_-$ merge into a single CONNECTED one. An analogous example to Poincare patches is the spherical surface cut into two halves and connected by the equator. This illustrates why Cauchy-Cauchy condition (4.83) is illegal on the AdS_5 boundary: on the "equator" connecting two Poincare patches, this condition fixes all of A^i and all of $\partial_5 A^i$, thus over-restricts the system. Furthermore, the choice for "equator" is just a global coordinate choice, we have an infinite number of ways to do it. The over-determination of boundary conditions is not removable.

4.7.2 The Consistency Constraint

A natural constraint to avoid over-determination in (4.82) is

$$g_{I\hat{J}^*}|_{z=z_-} = 0 \ , \tag{4.84}$$

which corresponds to a constraint on the reduced Kähler sub-manifold:

$$g_{I\hat{J}^*}|_{A^{\hat{K}}=c^{\hat{K}}} = 0 \ . \tag{4.85}$$

Before further discussion, let us clarify a few points about (4.85). First, this condition is stronger than $g_{I\hat{J}^*} = 0$ at the point $p : (A^i = c^i)$ [2], but is weaker than $g_{I\hat{J}^*} = 0$ everywhere[3]. Locally (4.85) defines an algebraic variety, i.e. it restricts the possible set of $c^{\hat{I}}$ we can choose as the boundary value of $A^{\hat{I}}$.

The constraint (4.85) has a clear physics implication. It means there is no kinetic term mixing A^I set and $A^{\hat{J}}$ set on the boundary. Choosing a coordinate system $(A^I, A^{\hat{J}})$ satisfying (4.85) corresponds to diagonalizing the sigma model properly on the boundary. Physicists are familiar with block diagonalizing or even diagonalizing kinetic term. We always admit such preferred choice when we analyze the detailed physics of a model.

If (4.85) is true on the complex n-dimension sub-manifold $\mathcal{S} : \{A^{\hat{I}} = c^{\hat{I}}\}$, the component boundary conditions (4.82) become

$$\partial_5 A^I| = iX^I| \ . \tag{4.86}$$

[2]This is a coordinate choice.
[3]This is a constraint on the hyper-Kähler manifold

which as Neumann conditions on A^I, are legal.

However, sigma model is based on a geometric language. From a geometric point of view, (4.85) is coordinate dependent. A change of coordinates may turn a "legal" boundary condition into a "forbidden" one. This makes no geometric sense. For physics application, we should define what we mean by **proper**, **allowed** and **preferred** mathematically. Then in practice, we can always pick a **preferred** choice.

To restore the coordinate independence, we have to "sum over" all possible choices of coordinates. The geometric proper condition then has an abstract form as follows:

Definition 1. (A^i, g, Ω) *is called a* **preferred** *Darboux coordinate system on a Darboux patch of hyper-Kähler manifold, if the following conditions are satisfied:*

1. *The symplectic structure Ω_{ij} takes the canonical form.*
2. *The following condition is true within the Darboux patch:*

$$g_{I\hat{J}^*}|_{A^{\hat{K}}=c^{\hat{K}}} = 0 \ . \tag{4.85}$$

Definition 2. *For any point $(A^I = c^I, A^{\hat{J}} = c^{\hat{J}})$ within a Darboux patch, the following boundary conditions are called* **separated** *and* **on-shell proper** *in the* **preferred** *Darboux coordinate system (A^i, g, Ω):*

$$A^{\hat{I}}| = c^{\hat{I}} \tag{4.39}$$

$$H_I| = 0 \ . \tag{4.40}$$

Under the coordinate transformation $(A^i, g) \rightarrow (A'^i, g')$*, the images of conditions (4.39) (4.40) are called* **on-shell proper** *in the* **allowed** *coordinates* (A'^i, g', Ω')*.*

Theorem 1. *The supersymmetric nonlinear sigma model in* AdS_5 *requires its boundary values to be* **on-shell proper***. Conversely, all* **on-shell proper** *values can be its boundary values.*

Theorem 2. On-shell proper *condition in any* **allowed** *coordinate system (not necessarily Darboux) is equivalent to the following condition:*

$$\Big(H_i dA^i\Big)\Big| = 0 \ , \tag{4.87}$$

where the symbol "$|$*" means on the n-d sub-manifold* $\mathcal{S}$*, the image of (4.39)*

This approach solves the problem formally. However, it seems to have less practical use. For instance, a general coordinate transformation may not preserve Ω_{ij} in the canonical form, therefore the boundary conditions may not be in a separated form. Only based on (4.87), we can not tell which n degrees of freedom are restricted.

We can reverse the question to seek a more practical method. Suppose at a point $p : (A^i = c^i)$ in a Darboux patch, one **preferred** Darboux coordinate is already known, can we then derive all the **allowed** coordinate sets that preserve Ω_{ij}, and permit **separated** boundary conditions like $A^{\hat{I}}| = c^{\hat{I}}$? More precisely, can we find all such **allowed** metrics g_{ij^*} at that point?

We can study a general diffeomorphism

$$\delta A^i = \xi^i \ . \tag{4.88}$$

Then after pulling back to p, all **allowed** metrics g can be generated from the **preferred** metric g_0. The first nontrivial constraint is that the symplectic structure Ω_{ij} must be preserved. This is the symplectic condition we mention earlier:

$$\delta_\xi \Omega = 0 \ . \tag{4.89}$$

More explicitly, we have

$$\Omega^{ij}\nabla_j \xi^k \Omega_{kl} + \nabla_l \xi^i = \Omega^{ij}\partial_j \xi^k \Omega_{kl} + \partial_l \xi^i = 0 \ . \tag{4.90}$$

Note, the relation (4.19) has been used.

The ξ here are not necessarily Killing. Actually, any isometry will preserve a **preferred** frames, for $\delta_\kappa g_{ij^*} = 0$. It can be shown that all vectors satisfying (4.89) generate a Lie group $\mathscr{S}$, while all tri-holomorphic Killing vectors generate the subgroup $\mathscr{H}$ of $\mathscr{S}$. One might consider the set of diffeomorphisms preserving Ω and the condition $g_{I\hat{J}^*}| = 0$. This set, unfortunately, turns out to have no group structure.

As the next step, we restrict ourselves to the subset of $\mathscr{S}$ that preserves n tangent vectors $\partial/\partial A^{\hat{I}}$, the condition is

$$\partial_L \xi^{\hat{I}} = \partial_{\hat{J}} \xi^{\hat{I}} = 0 \ . \tag{4.91}$$

Such ξ generate a subgroup of $\mathscr{S}$, denoted as $\mathscr{S}_0$. We may require a weaker condition as $\partial_L \xi^{\hat{I}} = 0$, so that $\{A^{\hat{I}}\}$ can mix among themselves while $A'^{\hat{I}} = c'^{\hat{I}}$ is still a good boundary condition. It can be shown that such ξ form another group $\mathscr{S}_s$. The following group relation is obvious:

$$\mathscr{H} \subset \mathscr{S}_0 \subset \mathscr{S}_s \subset \mathscr{S} \tag{4.92}$$

Thus we reach an algebraic result:

Theorem 3. *At any given point, all* **preferred** *Darboux coordinate systems are invariant under the group $\mathscr{H}$; all* **allowed** *Darboux coordinate systems are related by the group $\mathscr{S}$; all* **allowed** *Darboux coordinate systems that permit the same* **separated** *boundary conditions are related by the group $\mathscr{S}_0$; all* **allowed** *Darboux coordinate systems that permit the same type of* **separated** *boundary conditions are related by the group $\mathscr{S}_s$.*

When all vectors ξ are linear, the $2n \times 2n$ derivative matrix $(K)^i{}_j = \partial_j \xi^i$ reduces to a constant one. The condition (4.90) then becomes

$$\Omega^T K \Omega = -K \ , \tag{4.93}$$

which is the well-known symplectic condition. In this special case, group $\mathscr{S}_0$ corresponds to the following transformation:

$$\begin{pmatrix} A'^I \\ A'^{\hat{I}} \end{pmatrix} = \begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^I \\ A^{\hat{I}} \end{pmatrix} \ , \tag{4.94}$$

where $M = M^T$.

$\mathscr{S}_s$ corresponds to

$$\begin{pmatrix} A'^I \\ A'^{\hat{I}} \end{pmatrix} = \begin{pmatrix} (N^{-1})^T & M \\ 0 & N \end{pmatrix} \begin{pmatrix} A^I \\ A^{\hat{I}} \end{pmatrix}, \tag{4.95}$$

where $N^T M = M^T N$.

Both matrices in (4.7.2) and (4.7.2) are well known subgroups of the symplectic group. Thus $\mathscr{S}$ is the generalization of $Sp(2n, C)$. We should not be surprised, since A actually plays the role of q and $\partial_5 A$ plays the role of the momentum p. All potentially allowed boundary conditions should be enclosed by such symplectic reparameterizations.

In physics applications, it is important to find the preferred Darboux coordinates at any given point. However, because the Kähler metric appears in the condition (4.85), the curvature of the manifold is the obstruction for such coordinate choices. In general the preferred Darboux coordinates does not exist everywhere. Hence not all sub-manifolds can be used for a sigma model's boundary values. In general we have to rely on finding the variety in some given coordinate system and then performing coordinate transformations from there.

To illustrate this procedure, we discuss some examples here.

4.7.3 Examples

First we look at a concrete question: in a preferred frame, when will the boundary conditions $A^{\hat{I}} = 0$ be proper?

The bulk Kähler potential can be worked out locally as a Taylor expansion:

$$K = \sum_{n,m,p,q>0} C_{n_I,m_J,p_K,q_L} (A^I)^{\{n\}} (A^{*I^*})^{\{m\}} (A^{\hat{K}})^{\{p\}} (A^{*\hat{L}^*})^{\{q\}} \ , \tag{4.96}$$

where

$$(A^I)^{\{n\}} \equiv \prod_{\sum i_I = n} (A^I)^{i_I} \ . \tag{4.97}$$

(4.85) then becomes the following restriction (after removing the lowest components in the expansion via a Kähler transformation):

$$C_{n,m,0,1} = C_{n,m,1,0} = 0 \ . \tag{4.98}$$

When this condition is satisfied, such a hyper-Kähler manifold allows $A^{\hat{I}} = 0$ to be proper boundary conditions.

One can generalize this method to any given point.

Flat $2n$-d Complex Plane

In this case, (4.85) is satisfied at any point. The Cartesian coordinates clearly define a preferred frame where a single Darboux patch covers everything. All possible complex values thus can appear in the boundary conditions $A^{\hat{I}} = c^{\hat{I}}$.

Hyper-Kähler Cone

In most of the literature, the complex structure of the hyper-Kähler cone is not in the canonical form.

Fortunately, for a hyper-Kähler cone with complex dimension 2, a global transformation may map Ω into the canonical form, for instance

$$\zeta^1 = \frac{1}{2}u \tag{4.99}$$

$$\zeta^2 = \mathrm{e}^{2z}. \tag{4.100}$$

The Kähler metric in these new coordinates is

$$\begin{pmatrix} g_{11^*} & g_{12^*} \\ g_{21^*} & g_{22^*} \end{pmatrix} = \begin{pmatrix} 4\sqrt{\zeta^2\bar{\zeta}^2} & 2\bar{\zeta}^1\sqrt{\bar{\zeta}^2/\zeta^2} \\ 2\zeta^1\sqrt{\zeta^2/\bar{\zeta}^2} & (\frac{1}{4}+\zeta^1\bar{\zeta}^1)/\sqrt{\zeta^2\bar{\zeta}^2} \end{pmatrix} . \tag{4.101}$$

The constraint (4.85) in this case becomes

$$\zeta^1\sqrt{\frac{\bar{\zeta}^2}{\zeta^2}} = 0 \ , \tag{4.102}$$

with the only solution as $\zeta^1 = 0$. So only at this point, this coordinate choice is a preferred one.

But this is not the end of story. Now that the Darboux patch has covered the full complex manifold, we may expect the coordinates (ζ^1, ζ^2) to be an **allowed** one, although not **preferred**. In principle, one should study all possible coordinate transformations, or acting on this frame with $\mathscr{S}$. The situation turns out to be much simpler, because the following global transformation converts the cone into a flat 2-d complex plane:

$$y^1 = 2\zeta^1\sqrt{\zeta^2} = u\mathrm{e}^z \tag{4.103}$$

$$y^2 = \sqrt{\zeta^2} = \mathrm{e}^z\ . \tag{4.104}$$

Thus possible boundary conditions are $y^1 = \text{const}$ or $y^2 = \text{const}$.

In the next two examples, due to complicity, we will only solve the variety in the preferred frame. Lacking the clear existence of the essential Killing vector X, we can not apply them to AdS_5 sigma model at all. We only use them as boundary problem examples.

Eguchi-Hanson Model

This model has a target space as $T^*CP(1)$. Ω_{ab} is in canonical form globally, and the metric takes the form as

$$g_{ab^*} = \frac{1}{\rho^4\sqrt{1+\rho^4}}\begin{pmatrix} \rho^6 + Y\bar{Y} & -Y\bar{X} \\ -X\bar{Y} & \rho^6 + X\bar{X} \end{pmatrix}\ . \tag{4.105}$$

Obviously, this coordinate system is a preferred one on the sub-manifold $X = 0$ or $Y = 0$. It corresponds to two natural choices of boundary values.

C-Map

The C-map is a method to construct a $2n$-d hyper-Kähler manifold from a n-d Kählerian one with dimension. Ω_{ab} is in canonical form globally, while the Kähler metric is

$$g_{a\bar{b}} = \begin{pmatrix} (G+SG^{-1}\bar{S})_{A\bar{B}} & (-SG^{-1})_{A}{}^{\bar{B}} \\ (-G^{-1}\bar{S})^{A}{}_{\bar{B}} & (G^{-1})^{A\bar{B}} \end{pmatrix} , \tag{4.106}$$

where the n-d Kähler metric $G(X,\bar{X})$ can be related to a holomorphic potential $F(X)$ as

$$G_{A\bar{B}} = F_{AB} + \bar{F}_{\bar{A}\bar{B}} \ ; \tag{4.107}$$

and the matrix S depends on G, F and n new coordinate Y_I as

$$S_{IJ} = F_{IJK}G^{K\bar{L}}(Y+\bar{Y})_L \ . \tag{4.108}$$

The coordinate system is preferred when

$$S = 0 \Leftrightarrow Y_I + \bar{Y}_I = 0 \ . \tag{4.109}$$

This corresponds to a n-dimension real space. All values on it can be taken as the sigma model boundary values.

4.7.4 Geometric Relations in the Preferred Frame

In this section we will derive a set of relations on the geometric quantities in the preferred Darboux coordinates. These relations will be useful when we apply them to reduce the SUSY transformations near the boundary later.

- The metric has a block diagonalized form

$$g_{I\hat{J}^*}| = g_{\hat{J}I^*}| = g^{I\hat{J}^*}| = g^{\hat{J}I^*}| = 0 \tag{4.110}$$

$$g_{IJ^*}|g^{KJ^*}| = \delta_I^{J^*}, \qquad g_{\hat{I}\hat{J}^*}|g^{\hat{K}\hat{J}^*}| = \delta_{\hat{I}}^{\hat{J}^*} , \tag{4.111}$$

and the induced Kähler metric on the sub-manifold $\mathcal{S} : \{A^I, A^{\hat{J}} = 0\}$ is

$$h_{IJ^*} = g_{IJ^*}| . \tag{4.112}$$

- Since $g_{I\hat{J}^*}|_{A^{\hat{K}}=0} = 0$, differentiating it along the sub-manifold $\mathcal{S}$ gives

$$\partial_K g_{I\hat{J}^*}| = 0 . \tag{4.113}$$

However, the derivative normal to $\mathcal{S}$ in general is un-restricted:

$$\partial_{\hat{P}} g_{I\hat{J}^*}| = \partial_I g_{\hat{P}\hat{J}^*}| . \tag{4.114}$$

As a result, the following Chirstoffel symbols vanish:

$$\Gamma^{\hat{I}}_{JK}| = g^{\hat{I}\hat{P}^*}\partial_J g_{K\hat{P}^*}| = 0 \tag{4.115}$$

$$\Gamma^{I}_{J\hat{K}}| = g^{IL^*}\partial_J g_{\hat{K}L^*}| = 0 . \tag{4.116}$$

We can also use relation (4.19) to show:

$$\Gamma^{\hat{P}}_{\hat{K}\hat{L}}| = -\Omega^{\hat{P}J}\Gamma^{I}_{J\hat{K}}|\Omega_{I\hat{L}} = 0 . \tag{4.117}$$

Only the following Chirstoffel symbols can be nonzero:

$$\Gamma^{I}_{JK}|, \Gamma^{\hat{L}}_{\hat{P}K}|, \Gamma^{I}_{\hat{P}\hat{Q}}| \ .$$

Especially, we have

$$\Gamma^{I}_{JK}| = g^{IP^*}|\partial_J g_{KP^*}| \ , \tag{4.118}$$

which has the same value as the induced Chirstoffel symbol γ^{I}_{JK} on $\mathcal{S}$.

- One can further work out Riemann curvatures:

$$R_{\hat{I}J^*KL^*}| = R_{\hat{I}\hat{J}^*\hat{K}L^*}| = 0 \ . \tag{4.119}$$

On there other hand,

$$R_{IJ^*KL^*}| = g_{IP^*}|\partial_K \Gamma^{P^*}_{J^*L^*}| \ , \tag{4.120}$$

which equals to the induced curvature $r_{IJ^*KL^*}$.

The two other potentially non-zero curvatures are $R_{\hat{I}\hat{J}^*\hat{K}\hat{L}^*}|$ and $R_{IJ^*\hat{K}\hat{L}^*}|$

- There is a simple thumb rule: all metrics, Chirstoffel symbols and Riemann curvatures with odd number of hatted indices vanish in the preferred frame.

As a simple application of this rule, we have

$$R_{I\hat{J}^*}| = 0 \ , \tag{4.121}$$

which is consistent with the Ricci-flatness of the hyper-Kähler manifold.

- The other two Ricci-flat conditions become constraints:

$$R_{IJ^*}| = R_{IJ^*KL^*}|g^{KL^*}| + R_{IJ^*\hat{K}\hat{L}^*}|g^{\hat{K}\hat{L}^*} = 0 \tag{4.122}$$

$$R_{\hat{I}\hat{J}^*}| = R_{\hat{I}\hat{J}^*KL^*}|g^{KL^*}| + R_{\hat{I}\hat{J}^*\hat{K}\hat{L}^*}|g^{\hat{K}\hat{L}^*} = 0 \ . \tag{4.123}$$

  Note the induced Ricci tensor on the Kähler sub-manifold $\mathcal{S}$ is not necessarily zero:

$$r_{IJ^*} = R_{IJ^*KL^*}|g^{KL^*}| \ . \tag{4.124}$$

To conclude, we find that in the preferred frame, the condition (4.85) simplifies geometric relations. On the other hand, the induced Kähler geometry on the sub-manifold $\mathcal{S}$ can still be rich.

4.8 Gibbons-Hawking-York Term

4.8.1 Derivation from the Variational Principle

In this section we will derive the appropriate surface term for the component action. The philosophy is still the variational principle: we require the component action to produce proper equations of motion after imposing boundary conditions that we previously derived.

The relevant terms from a field variation on the rigid sigma action are

$$\Delta S = \int d^5x \mathrm{e}^{-4kz}\Big\{ - g_{ij^*}\partial_5(\Delta A^i)\partial_5(A^{*j^*}) + h.c. + ... \\ +\frac{i}{2}g_{ij^*}\bar{\Psi}^{j^*}\gamma^5\partial_5(\Delta\Psi^i) + \frac{i}{2}g_{ij^*}\Gamma^i_{kl}\bar{\Psi}^{j^*}\gamma^5\Psi^k\partial_5(\Delta A^l)\Big\} . \quad (4.125)$$

This generates the following surface term after integration by parts:

$$S.T. = \int d^4x \mathrm{e}^{-4kz}\Big\{ - g_{ij^*}\Delta A^i\partial_5 A^{*j^*} + h.c. \\ +\frac{i}{2}g_{ij^*}\bar{\Psi}^{j^*}\gamma^5\Delta\Psi^i + \frac{i}{2}g_{ij^*}\Gamma^i_{kl}\bar{\Psi}^{j^*}\gamma^5\Psi^k\Delta A^l\Big\} . \quad (4.126)$$

Expanding in 2-component spinors, this surface term becomes

$$S.T. = \int d^4x \mathrm{e}^{-4kz}\Big\{ - g_{IJ^*}\Delta A^I\partial_5 A^{*J^*} - g_{\hat{I}\hat{J}^*}\Delta A^{\hat{I}}\partial_5 A^{*\hat{J}^*} \\ -g_{I\hat{J}^*}\Delta A^I\partial_5 A^{*\hat{J}^*} - g_{\hat{I}J^*}\Delta A^{\hat{I}}\partial_5 A^{*J^*} + h.c. \\ +\frac{1}{2}\Omega_{ij}\chi^i\Delta\chi^j + \frac{1}{2}\Omega_{p^*q^*}\bar{\chi}^{p^*}\Delta\bar{\chi}^{q^*} \\ +\frac{1}{2}\Omega_{ip}\Gamma^i_{kl}\chi^p\chi^k\Delta A^l - \frac{1}{2}\Omega_{s^*j^*}\Gamma^{s^*}_{q^*k^*}\bar{\chi}^{j^*}\bar{\chi}^{k^*}\Delta A^{*q^*}\Big\} \quad (4.127)$$

We work in the Darboux patch and impose the following boundary conditions:

$$A^{\hat{I}}| = c^{\hat{I}} \quad (4.38)$$

$$g_{I\hat{J}^*}| = 0 \; . \quad (4.85)$$

Requiring (4.38) to be $\mathcal{N} = 1$ invariant, we find the following secondary conditions:

$$\chi^{\hat{I}}| = 0 \quad (4.128)$$

$$\partial_5 A^I| = iX^I| \; . \quad (4.129)$$

Here we have used geometric relations derived in Sec. 4.7.4. Then the surface term (4.126) can be simplified as

$$\begin{aligned} S.T. &= \int d^4x\mathrm{e}^{-4kz}\Big\{ig_{IJ^*}\bar{X}^{J^*}\Delta A^I + h.c.\Big\} \\ &= -\Delta\Big\{\int d^4x\mathrm{e}^{-4kz}\mathrm{D}^{(X)}\Big\}\ . \end{aligned} \tag{4.130}$$

This variation can be canceled by the following boundary-localized term

$$\int d^5x\mathrm{e}^{-4kz}Y = \int d^4x\mathrm{e}^{-4kz}\mathrm{D}^{(X)}\ . \tag{4.131}$$

This is the Gibbons-Hawking-York term [36–38] for hypermultiplets.

4.8.2 Y-term and the Component Action

Now we can reproduce the component action from $\mathcal{N}=1$ superspace. Using boundary condition (4.38) and integrating out all auxiliary fields $\mathcal{F}^i$, we obtain the following terms:

$$\begin{aligned} S &= \int d^4xdze^{-4kz}-g_{ij^*}\mathcal{F}^i\mathcal{F}^{*j^*}+... \\ &= \int d^4xdze^{-4kz}\left\{-g_{ij^*}\left(\frac{1}{2}\Gamma^i_{pq}\chi^p\chi^q-\Omega^i{}_{k^*}\partial_5A^{*k^*}-i\Omega^i{}_{k^*}\bar{X}^{k^*}\right)\right. \\ &\qquad\qquad \left.\cdot\left(\frac{1}{2}\Gamma^{j^*}_{m^*n^*}\bar{\chi}^{m^*}\bar{\chi}^{n^*}-\Omega^{j^*}_{\ l}\partial_5A^l+i\Omega^{j^*}_{\ l}X^l\right)\right\}+... \\ &= \int d^4xdz\mathrm{e}^{-4kz}\left\{-g_{ij^*}\partial_5A^i\partial_5A^{*j^*}-g_{ij^*}X^i\bar{X}^{j^*}-ig_{ij^*}\partial_5A^i\bar{X}^{j^*}+ig_{ij^*}\partial_5A^{*j^*}X^i\right\}+... \\ &= \int d^4xdz\mathrm{e}^{-4kz}\left\{-g_{ij^*}\partial_5A^i\partial_5A^{*j^*}-g_{ij^*}X^i\bar{X}^{j^*}+\partial_5\mathrm{D}^{(X)}\right\}+... \end{aligned} \tag{4.132}$$

One more integration by parts puts the action in a more standard form:

$$\begin{aligned} S &= \int d^4x dz \mathrm{e}^{-4kz}\Big\{ - g_{ij^*}\partial_5 A^i \partial_5 A^{*j^*} - \Big(g_{ij^*} X^i \bar{X}^{j^*} - 4k\mathrm{D}^{(X)} \Big) \Big\} + ... \\ & \quad + \int d^4x \mathrm{e}^{-4kz} \mathrm{D}^{(X)}|_{z_-}^{z_+} \ . \end{aligned} \tag{4.133}$$

We see that the component action has a scalar potential as

$$\mathcal{V} = g_{ij^*} X^i \bar{X}^{j^*} - 4k\mathrm{D}^{(X)} \ . \tag{2.31}$$

To precisely match the superspace action, the surface term in (4.133) must be added by hand to the component action. This is just the Gibbons-Hawking-York term we found previously:

$$Y = [\delta(z - z_+) - \delta(z - z_-)]\mathrm{D}^{(X)} \ , \tag{4.134}$$

which has a natural interpretation as a boundary-localized Killing potential.

This term also exists in the $k \to 0$ limit. It is not supersymmetric, so while the superspace action is $\mathcal{N} = 1$ SUSY invariant, the bulk component action without the Y-term is not. This situation is similar to the vector sector Y-term (3.118).

In the gauged model, both (4.134) and (3.118) are required, and the Y-term is simply a sum of them.

4.8.3 Y-term and the Gauge Constraint

Now we can understand how adding the Y-term changes the gauged sigma model's constraint (3.29). Under the global transformation δ_T, the changing of the combined action is

$$
\begin{aligned}
& \int d^4xdz\mathrm{e}^{-4kz}(-\delta_T\mathcal{V}+\delta_T Y) \\
= & +\int d^4xdz\mathrm{e}^{-4kz}\left[4k(T^i\mathrm{D}_i+\bar{T}^{j^*}\mathrm{D}_{j^*})\right]+\int d^4xdz\partial_5[\mathrm{e}^{-4kz}(T^i\mathrm{D}_i+\bar{T}^{j^*}\mathrm{D}_{j^*})] \\
= & +\int d^4xdz\mathrm{e}^{-4kz}4k\partial_5(T^i\mathrm{D}_i+\bar{T}^{j^*}\mathrm{D}_{j^*}) \\
= & -\int d^4xdz\mathrm{e}^{-4kz}4ik\partial_5(T^iX_i-X^iT_i)\ . \qquad (4.135)
\end{aligned}
$$

Now, we don't need the strong constraint (3.29) any more. The condition $[X,T]=0$ is enough to guarantee $T^iX_i - X^iT_i = \mathrm{const}$ so that the total action is global invariant.

It is well known that Y-term can change the boundary condition. It is interesting to find that in warped space Y-term may also change constraints in the bulk.

4.9 General Boundary Conditions

4.9.1 Geometric Picture

The boundary conditions $A^{\hat{I}}| = c^{\hat{I}}$ assign constant values to n scalar fields on a 4-d plane. Moving around this boundary, these values do not change, so they should be viewed as 4-d VEVs. This is a global reduction: on any point on the plane, unrestricted boundary fields A^I live on the same n-dimensional Kähler sub-manifold.

In the previous section, we have shown that the superspace action without boundary coupling gives such a global reduction naturally. However, to include boundary sources in the model, and to have a proper path integral quantization scheme, the boundary values of $A^{\hat{I}}$ should be generalized to fixed values instead of constant ones. Physically this means we should allow fluctuations of $A^{\hat{I}}$ around their VEVs. These fluctuations are not dynamical ones, on each space-time boundary point they must be fixed, so that $\Delta A^{\hat{I}} = 0$. For practical purpose they should also be small, maybe even controlled by the warp factor, so that 4-d Lorentz symmetry is preserved. Since the $A^{\hat{I}}$ change from point to point, this type of general boundary condition corresponds to a more complicated geometric picture: on the boundary, the reduced sub-manifold is not a rigid one any more. Although on each space-time point we still have a constant n-dimensional Kähler sub-manifold, moving around the space-time boundary induces its continuous vibration. There is a good geometric reason for this vibration to be small too. If the sub-manifold has a finite deformation instead, we will face a non-trivial topological issue: when one walks around a closed loop in the 4-d boundary, how can the induced deformation always returns the n-d Kähler manifold back to its initial shape?

In this section, we will discuss how to derive these general boundary conditions from the variational principle.

Previously we have worked out $\mathcal{N} = 1$ boundary conditions in superspace. Later while comparing the superspace action to the component expression, we found a Y-term is necessary as the additional boundary term to the component bulk action. This Y-term is not supersymmetric. It plays two roles, firstly it changes the boundary variation to a more proper form, secondly it makes the component action truly $\mathcal{N} = 1$

invariant. It is then natural to extend this idea: maybe $\mathcal{N}=1$ superspace action can also have a $\mathcal{N}=1$ but not $\mathcal{N}=2$ invariant Y-term. It may help us to obtain general boundary conditions.

4.9.2 Vector Sector

First, as a warm-up, let us discuss the $\mathcal{N}=1$ Y-term in the vector sector. It plays an interesting role. In $U(1)$ gauge case, a field variation produces the following surface term:

$$S.T. = \int d^4x \int d^4\theta \mathrm{e}^{-2kz} 2(\chi + \bar{\chi} - \partial_5 V)\Delta V \ . \tag{4.136}$$

Besides the known constant condition $(\chi + \bar{\chi} - \partial_5 V)| = 0$, the only general boundary condition can be derived is the Dirichlet type (on V):

$$\Delta V| = 0 \ . \tag{4.137}$$

To get the general Neumann condition, we add a $\mathcal{N}=1$ Y-term to the bulk action:

$$\int d^4x \int d^4\theta \mathrm{e}^{-2kz} Y = -\int d^4x \int d^4\theta \mathrm{e}^{-2kz} 2(\chi + \bar{\chi} - \partial_5 V)V \ . \tag{4.138}$$

Then surface term becomes

$$S.T.' = \int d^4x \int d^4\theta \mathrm{e}^{-2kz} - 2\Delta(\chi + \bar{\chi} - \partial_5 V)V \ . \tag{4.139}$$

The vanishing of this requires a Neumann boundary condition (on V):

$$\Big[\Delta(\partial_5 V) - \Delta\chi - \Delta\bar{\chi}\Big]\Big| = 0 \ . \tag{4.140}$$

To conclude, the $\mathcal{N} = 1$ Y-term converts one type of boundary condition to another. This result is already well known at the component level [51].

4.9.3 Rigid Sigma Model

Now we return to the rigid sigma model in AdS_5. The field variation of the bulk action in superspace is:

$$\Delta S = \int d^5x \mathrm{e}^{-3kz} \int d^2\theta H_i \partial_5 \Delta(\Phi^i) + h.c. + ... \tag{4.141}$$

It gives the following surface term after integration by parts

$$S.T. = \int d^4x \mathrm{e}^{-3kz} \int d^2\theta H_i \Delta\Phi^i + h.c. + ... \tag{4.142}$$

Previously, we have worked in a Darboux patch and found the following geometric reduction

$$\Phi^{\hat{I}}| = c^{\hat{I}} \ . \tag{4.38}$$

On the boundary, this gets rid of half of superfield degrees of freedom completely.

Boundary conditions (4.38) are valid because we can always find the proper gauge transformation $H_i \longrightarrow H'_i$ so that $H'_I| = 0$.

Now we can stay in the Darboux patch and seek the following general boundary conditions on hypermultiplets:

$$\Delta\Phi^{\hat{I}}| = 0 \ . \tag{4.143}$$

So n chiral superfiels $\Phi^{\hat{I}}$ are not necessarily constants on the space-time boundary ($\partial_m \Phi \neq 0$ in general), but are rather kept as n specified functions $f^{\hat{I}}(x^m)$. Obviously this type of general boundary conditions does not always admit solutions to $H_I| = 0$. So in general $H_i \Delta\Phi^i$ does not vanish on the boundary.

We can fix this problem by adding a $\mathcal{N} = 1$ Y-term with the following form:

$$\int d^4x \int d^2\theta \mathrm{e}^{-3kz} Y = -\int d^4x \mathrm{e}^{-3kz} H_I \Phi^I + h.c. \ , \tag{4.144}$$

where the summation runs only from $I = 1$ to n instead of $2n$.

With this Y-term, the combined action generates the following surface term under a field variation:

$$S.T. = -\int d^4x \int d^2\theta \mathrm{e}^{-3kz} \Delta H_I \Phi^I + h.c. \ . \tag{4.145}$$

The constraint on H_I seems to be:

$$\partial_J H_I| = 0 \ . \tag{4.146}$$

However, the situation is trickier now, because the Y-term we add is not "gauge" invariant. The field redefinition freedom is lost[4].

The appropriate approach is to add a surface term and study field variation of the combined action carefully:

$$S' = S + \int Y + \int d^4x \Big[\mathrm{e}^{-3kz} P + h.c. \Big] \Big|_{z_-}^{z_+}$$

$$\Rightarrow S.T.' = \int d^4x \mathrm{e}^{-3kz} \Big[- H_{I,J} \Delta\Phi^J \Phi^I + \partial_J P \Delta\Phi^J \Big] \Big| + h.c. \ . \tag{4.147}$$

[4]We can also see this from the fact that adding boundary term P only induces redefinition on the bulk term but not on Y-term.

To have vanishing surface term, we requires

$$\partial_I P| = H_{J,I}\Phi^J| \ . \tag{4.148}$$

Fortunately, due to $H_{I,J} = H_{J,I}$ on the Darboux patch, the integrability condition for P is satisfied automatically[5], thus (4.146) is still accessible.

To conclude, we find that adding the Y-term (4.144) produces the general boundary conditions (4.143) from the variational principle.

4.9.4 Reduction to Component Formalism

To further study the consistency of (4.143), we will reduce it to components. First, we have to solve auxiliary fields $\mathcal{F}^i$.

After integration by parts, the component action involving becomes

$$\begin{aligned} S \;\; = \;\; & \int d^5x \mathrm{e}^{-4kz}\Big\{ g_{ij^*}\mathcal{F}^i\mathcal{F}^{*j^*} - \frac{1}{2}g_{ij^*}\Gamma^{j^*}_{k^*l^*}\bar{\chi}^{k^*}\bar{\chi}^{l^*} - \frac{1}{2}g_{ij^*}\Gamma^i_{pq}\bar{\mathcal{F}}^{j^*}\chi^p\chi^q \\ & \qquad + \Big[H_{i,j}\mathcal{F}^j\partial_5 A^i + G_i\mathcal{F}^i + h.c.\Big] + ...\Big\} + \\ & + \int d^5x \mathrm{e}^{-3kz} H_i \partial_5\Big(\mathrm{e}^{-kz}\mathcal{F}^i\Big) + h.c. + ... \end{aligned}$$

[5] Formula (4.148) can be understood in an interesting way, if we rewrite P as

$$P \equiv -Y + \Pi = H_I\Phi^I + \Pi \ . \tag{4.149}$$

The first term kills Y-term and the second term Π can induce a usual bulk field redefinition. The boundary problem with Y-term then reduces to the problem without Y-term. From (4.142), it is easy to read off the proper gauge fixing condition:

$$H'_I| = (H_I + \partial_I\Pi)| = 0 \ . \tag{4.150}$$

This is analogous to the Stueckelberg trick.

$$
-\int d^4x \mathrm{e}^{-4kz}\Big(H_I\mathcal{F}^I + H_{I,\hat{J}}\mathcal{F}^{\hat{J}}A^I + h.c. + ...\Big)\Big|_{z_-}^{z_+}
$$

$$
\begin{aligned}
= \quad & \int d^5x \mathrm{e}^{-4kz}\Big\{g_{ij^*}\mathcal{F}^i\mathcal{F}^{*j^*} - \frac{1}{2}g_{ij^*}\Gamma^{j^*}_{k^*l^*}\bar{\chi}^{k^*}\bar{\chi}^{l^*} - \frac{1}{2}g_{ij^*}\Gamma^i_{pq}\bar{\mathcal{F}}^{j^*}\chi^p\chi^q \\
& \qquad + \Big[\Omega_{ij}\mathcal{F}^j\partial_5A^i + (G_i + 3kH_i)\mathcal{F}^i + h.c.\Big] + ...\Big\} \\
& + \int d^5x \mathrm{e}^{-4kz}\Big(H_{\hat{I}}\mathcal{F}^{\hat{I}} - H_{I,\hat{J}}\mathcal{F}^{\hat{J}}A^I + h.c.\Big)\Big[\delta(z-z_+) - \delta(z-z_-)\Big] \ . \qquad (4.151)
\end{aligned}
$$

We can solve $\mathcal{F}^i$ in the bulk as

$$
\mathcal{F}^I = \frac{1}{2}\Gamma^I_{jk}\chi^j\chi^k - \Omega^{I\hat{J}}g_{\hat{J}q^*}(\partial_5A^{*q^*} + i\bar{X}^{q^*}) \qquad (4.152)
$$

$$
\mathcal{F}^{\hat{I}} = \frac{1}{2}\Gamma^{\hat{I}}_{jk}\chi^j\chi^k - \Omega^{\hat{I}J}g_{Jq^*}(\partial_5A^{*q^*} + i\bar{X}^{q^*}) \ . \qquad (4.153)
$$

How about their solutions on the boundary? At the first sight, they seem to have boundary corrections because $\partial_k(H_{\hat{I}}\mathcal{F}^{\hat{I}} - H_{I,\hat{J}}\mathcal{F}^{\hat{J}}A^I) = H_{\hat{J},K} - H_{K,\hat{J}} = \Omega_{\hat{J}K} \neq 0$. However this is an illusion! The important fact is that we are doing fixed value variation now, thus the condition $\Delta\mathcal{F}^{\hat{I}}| = 0$ forbids any boundary contribution to the $\mathcal{F}$-EOMs. As a result, bulk solutions (4.152) and (4.153) can be extended continuously to the boundary.

At the component level, the following set of conditions can be reduced from(4.143):

$$
\Delta A^{\hat{I}} = 0 \qquad (4.154)
$$

$$
\Delta\chi^{\hat{I}} = 0 \qquad (4.155)
$$

$$
\Delta\mathcal{F}^{\hat{I}} = 0 \ , \qquad (4.156)
$$

where $\mathcal{F}^{\hat{I}}$ are n functions with physical field dependence as (4.153); (4.155) and (4.156) are consistency conditions for (4.154) to be $\mathcal{N}=1$ invariant.

The detail calculation is complicated, so we first work out the case when the hyper-Kähler manifold is flat.

Flat Kähler Case

Let us assume the metric g_{ij^*} has a constant but not necessarily unit matrix form. $\mathcal{F}^{\hat{I}}$ then has the following form:

$$\mathcal{F}^{\hat{I}} = -\Omega^{\hat{I}J} g_{Jq^*} (\partial_5 A^{*q^*} + i\bar{X}^{q^*}) \ . \tag{4.157}$$

The consistency condition (4.156) becomes

$$0 = \Omega^{\hat{I}J} g_{JL^*} \left[\Delta(\partial_5 A^{*L^*}) + i\bar{X}^{L^*}_{R^*} \Delta A^{*R^*} \right] \Bigg| + \Omega^{\hat{I}J} g_{J\hat{Q}^*} \left[\Delta(\partial_5 A^{*\hat{Q}^*}) + i\bar{X}^{\hat{Q}^*}_{R^*} \Delta A^{*R^*} \right] \Bigg| \ . \tag{4.158}$$

Here comes the trouble: while $A^{\hat{I}}$ get restricted, their derivative $\partial_5 A^{\hat{I}}$ should not be restricted.

- We first assume the following block diagonalization condition

$$g_{J\hat{Q}^*}| = 0 \ , \tag{4.159}$$

(4.158) then becomes

$$\Delta(\partial_5 A^L) - iX^L_R \Delta A^R = 0 \ . \tag{4.160}$$

An interpretation of (4.160) is needed. This condition only makes sense if we consider that $\partial_5 A^I$ become field dependent functions near the boundary:

$$\partial_5 A^I = iK^I(A^J) \ . \tag{4.161}$$

Physically this is possible when on-shell EOMs of A^i are used. (4.160) then restricts the induced n-dimensional holomorphic vector K as:

$$K^I_J = X^I_J| \ . \tag{4.162}$$

- Now we can go back to study the general case with $g_{J\hat{Q}^*}| \neq 0$.

  $\partial_5 A^I|$ in this case must be considered as functions of both A^I and $\partial_5 A^{\hat{J}}$:

$$\partial_5 A^I = i\Upsilon^I(A^J, \partial_5 A^{\hat{L}}) \ , \tag{4.163}$$

(4.156) then requires

$$g_{IJ^*}\frac{\partial}{\partial(\partial_5 A^{\hat{L}})}\Upsilon^I = ig_{\hat{L}J^*} \tag{4.164}$$

$$g_{IJ^*}\frac{\partial}{\partial A^R}\Upsilon^I = g_{qJ^*}X^q_R \ . \tag{4.165}$$

Because g_{ij^*} is a constant metric, hitting both sides of (4.164) with $\partial/\partial(\partial_5 A^{\hat{L}})$ reveals that Υ^I is only linear in $\partial_5 A^{\hat{L}}$. Thus we can always perform a coordinate transformation to make Υ^I independent of $\partial_5 A^{\hat{L}}$. This actually is the local diagonalization procedure to archive $g_{I\hat{J}^*}| = 0$. In Sec. 4.7.3, we have mentioned such a preferred coordinate system is always available, due to the flatness of the complex plane.

General Kähler Case

As the flat Kähler case, we again make the choice (4.159). Since A^I are kept as unrestricted, this condition locally means

$$\Gamma^{\hat{I}}_{JK}| = g^{\hat{I}\hat{P}^*}\partial_J g_{K\hat{P}^*}| = 0 \ . \tag{4.166}$$

n consistency conditions $\Delta\mathcal{F}^{\hat{I}} = 0$ then become

$$-\Omega^{\hat{I}J}\Delta\Big[g_{JP^*}\Big(\partial_5 A^{*P^*} + i\bar{X}^{P^*}\Big)\Big] + \Delta\Big(\Gamma^{\hat{I}}_{J\hat{K}}\chi^J\chi^{\hat{K}}\Big) = 0 \ . \tag{4.167}$$

Now we have to assume that near the boundary, $\partial_5 A^I$ are functions of both the vector X^I and fermions χ.

Using the Darboux compatible relation, we find a simple result:

$$\partial_5 A^I| = \Big(iX^I + g^{IJ^*}\Gamma^{L^*}_{J^*P^*}\overline{\Omega}_{L^*\hat{K}^*}\bar{\chi}^{J^*}\bar{\chi}^{\hat{K}^*}\Big)\Big| - g^{IQ^*}\Omega_{Q^*\hat{L}^*}\bar{f}^{\hat{L}^*} \ , \tag{4.168}$$

where constant functions $f^{\hat{I}}$ are boundary values of $F^{\hat{I}}$. They have to be small to avoid spontaneous breaking of supersymmetry.

To summarize, in general Kähler case, the boundary conditions have simple forms as Dirichlet-Neumann mixed type:

$$\Delta A^{\hat{I}}| = 0 \tag{4.154}$$

$$\Delta\chi^{\hat{I}}| = 0 \tag{4.155}$$

$$\partial_5 A^I| = \Big(iX^I + g^{IJ^*}\Gamma^{L^*}_{J^*P^*}\overline{\Omega}_{L^*\hat{K}^*}\bar{\chi}^{J^*}\bar{\chi}^{\hat{K}^*}\Big)\Big| - g^{IQ^*}\Omega_{Q^*\hat{L}^*}\bar{f}^{L^*} \ . \tag{4.168}$$

4.9.5 General Y-Term In Components

With proper boundary conditions (4.154, 4.155, 4.168) in hand, we can forget about the superspace method completely and study the variational principle in component language. The rigid sigma model in the bulk generates the following surface terms under a field variation:

$$\begin{aligned} S.T. \; = \; \int d^4x \mathrm{e}^{-4kz} \Big\{ & - g_{IJ^*}\Delta A^I \partial_5 A^{*J^*} - g_{\hat{I}\hat{J}^*}\Delta A^{\hat{I}} \partial_5 A^{*\hat{J}^*} \\ & - g_{I\hat{J}^*}\Delta A^I \partial_5 A^{*\hat{J}^*} - g_{\hat{I}J^*}\Delta A^{\hat{I}} \partial_5 A^{*J^*} + h.c. \\ & -\frac{1}{2}\Omega_{ij}\chi^i \Delta\chi^j + \frac{1}{2}\Omega_{p^*q^*}\bar{\chi}^{p^*}\Delta\bar{\chi}^{q^*} \\ & +\frac{1}{2}\Omega_{ip}\Gamma^i_{kl}\chi^p\chi^k\Delta A^l + \frac{1}{2}\Omega_{s^*j^*}\Gamma^{s^*}_{q^*k^*}\bar{\chi}^{j^*}\bar{\chi}^{k^*}\Delta A^{*q^*} \Big\} \,. \end{aligned} \tag{4.169}$$

Since both ΔA^I and $\Delta\partial_5 A^{\hat{J}}$ are unrestricted, the second line can not be removed by any Y-term, we have to stick with the constraint (4.159) to kill it. The third line, on the other hand, can be removed by a Y-term with an integrand $\Omega_{I\hat{J}}\chi^I\chi^{\hat{J}}$. This is precisely the boundary term Henningson and Sfetsos used to derive spinor AdS/CFT correspondence [52]. To remove the first and the last lines, we will use the consistency condition (4.168), and add a Y-term integrand as $-\Omega_{I\hat{J}}A^I f^{\hat{J}} - \Omega_{I^*J^*}A^{*I^*}\bar{f}^{\hat{J}^*} - \mathrm{D}^{(X)}$. In total, the component Y-term is as follows:

$$\int d^4x \mathrm{e}^{-4kz}\Big\{ - \Omega_{I\hat{J}}A^I f^{\hat{J}} - \Omega_{I^*J^*}A^{*I^*}\bar{f}^{\hat{J}^*} - \mathrm{D}^{(X)} + \Omega_{I\hat{J}}\chi^I\chi^{\hat{J}} + \Omega_{I^*\hat{J}^*}\bar{\chi}^{I^*}\bar{\chi}^{\hat{J}^*}\Big\} \,. \tag{4.170}$$

To summarize, in both superspace and the component formalism, we derive general boundary conditions (4.154, 4.155, 4.168) from the variational principle. Both formalism need additional boundary-localized Y-terms. It can be shown that the $\mathcal{N}=1$ Y-term in superspace is not invariant under the second SUSY. Hence we have to conclude that the general boundary conditions breaks $\mathcal{N}=2$ SUSY to $\mathcal{N}=1$ on the boundary.

4.10 Induced SUSY Transformation on the Boundary

In this section, we will study the SUSY transformations on the boundary. Because our boundary conditions are manifestly $\mathcal{N}=1$ invariant, the first SUSY transformation on the boundary will be just the same as in the bulk case. The only non-trivial question is: what happen to the second SUSY transformation on the boundary?

At the first sight, such a question seems to be illegal. A common understanding is that the presence of a boundary breaks the translation invariance normal to it. Since in both the flat and warped cases, the closure $[\delta_\epsilon, \delta_\eta]$ generate a translation along the fifth dimension, keeping the regulator branes at finite locations breaks the fifth translation and the second SUSY as well.

There turns out to be more story behind this argument. When we send $z_- \to -\infty$ and $z_+ \to \infty$ and recover the full AdS_5, any breaking on $z = z_+$ is compensated by the other Poincare patch. The $z = z_-$ brane is where we can evaluate the breaking effect.

Because the general $\mathcal{N}=1$ Y-term in Sec. 4.8 is not $\mathcal{N}=2$ invariant, "fixed-value" boundary conditions break the second SUSY. In this section we will only discuss the case with "constant-value" boundary conditions.

4.10.1 Vector Multiplet

There are two choices of boundary conditions. First we take the one breaks gauge symmetry:

$$V| = 0 \ . \tag{4.171}$$

Plugging this into the second SUSY transformation on chiral multiplet χ, we have

$$\delta_\eta \chi| = -k\eta_s^A D_A \chi| \ . \tag{4.172}$$

This matches the superconformal transformation written in $\mathcal{N}=1$ superspace.

Now we calculate the breaking effect on the boundary condition near the boundary. When $z=z_-$-brane is kept at a finial location, the boundary condition $V|=0$ is not invariant under the second SUSY:

$$\delta_\eta V| = (\chi| + \bar{\chi}| - \partial_5 V|)(\theta\eta + \bar{\theta}\bar{\eta}) \ . \tag{4.173}$$

Near the boundary, we can use the on-shell profiles for free component fields to evaluate this quantity. Since

$$v_m \sim v_m^{(1)}(x)\mathrm{e}^{0kz} + v_m^{(2)}(x)\mathrm{e}^{2kz} \tag{4.174}$$

$$\Sigma \sim \Sigma(x)\mathrm{e}^{2kz} \tag{4.175}$$

$$v_5 \sim v_5^{(1)}(x)\mathrm{e}^{2kz} + v_5^{(2)}(x)\mathrm{e}^{4kz} \ , \tag{4.176}$$

we have

$$\delta_\eta V| \sim \mathrm{e}^{2kz} f(x) \ . \tag{4.177}$$

This clearly shows that although the second SUSY breaks the boundary condition, the breaking effect is damped to 0 when we send $z_- \to -\infty$.

We then take the second choice of boundary conditions that preserves gauge symmetry:

$$(\chi| + \bar{\chi}| - \partial_5 V|) = 0 \ . \tag{4.178}$$

The second SUSY on the multiplet clearly becomes a special superconformal transformation:

$$\delta_\eta V| = -k\eta_s^A D_A V| \ . \tag{4.179}$$

Now we calculate the breaking effect on the boundary condition from the second SUSY:

$$\delta_\eta(\chi| + \bar{\chi}| - \partial_5 V|) = -\mathrm{e}^{2kz}\eta W| - \mathrm{e}^{2kz}\bar{\eta}\bar{W}| - k\eta_s^A D_A(\partial_5 V)| \ . \tag{4.180}$$

The first two terms are controlled by the warp factor, the last term is not. However using on-shell profiles we can see they are both damped to 0 when $z_- \to -\infty$.

So we feel confident to claim that both boundary conditions reduce the unrestricted boundary values to a $\mathcal{N} = 1$ superconformal multiplet.

4.10.2 Rigid Hypermultiplets

The second SUSY transformations on bulk hypermultiplets are

$$\delta\Phi^i = \frac{1}{2}e^{kz}\bar{D}^2\left[\Omega^{ij}K_j(\theta\eta+\bar{\theta}\bar{\eta})\right] - 12k\Omega^{ij}\left(H_j+\frac{G_j}{3k}\right)\theta\eta - k\eta_s^A D_A\Phi^i\ , \qquad (4.181)$$

and the boundary conditions are

$$\Phi^{\hat{I}}| = 0 \qquad (4.38)$$

$$H_I| = 0\ . \qquad (4.40)$$

We take the preferred Darboux coordinate system, i.e.

$$K_{J\hat{P}^*}| = 0\ . \qquad (4.182)$$

This is just the $\mathcal{N}=1$ invariant version of (4.85).

On the boundary, the second SUSY transformations on n unrestricted chiral superfields Φ^I become

$$\begin{aligned}
\delta_\eta\Phi^I| &= \frac{1}{2}\mathrm{e}^{kz}\bar{\Omega}^{I\hat{J}}D^2\left[K_{\hat{J}}(\theta\eta+\bar{\theta}\bar{\eta})\right]\Big| - 12k\Omega^{I\hat{J}}\left(H_{\hat{J}}+\frac{G_{\hat{J}}}{3k}\right)\Bigg|\theta\eta - k\eta_s^A D_A\Phi^I| \\
&= \frac{1}{2}\mathrm{e}^{kz}\bar{\Omega}^{I\hat{J}}\left[\bar{D}_{\dot{\alpha}}K_{\hat{J}\hat{P}^*}\bar{D}^{\dot{\alpha}}\bar{\Phi}^{\hat{P}^*} + 2K_{J\hat{P}^*}\bar{D}_{\dot{\alpha}}\bar{\Phi}^{\hat{P}^*}\bar{\eta}^{\dot{\alpha}} + K_{\hat{J}\hat{P}^*}\bar{D}^2\Phi^{\hat{P}}\right]\Big| \\
&\quad -12k\Omega^{I\hat{J}}\left(H_{\hat{J}}+\frac{G_{\hat{J}}}{3k}\right)\Bigg|\theta\eta - k\eta_s^A D_A\Phi^I| \\
&= -4i\ X^I\big|\,\theta\eta - k\eta_s^A D_A\Phi^I|\ . \qquad (4.183)
\end{aligned}$$

Then if we define the holomorphic vector

$$W^I = 6i\Omega^{I\hat{J}}\left(H_{\hat{J}}+\frac{G_{\hat{J}}}{3k}\right)\Bigg| = -\frac{2}{k}\ X^I\Big|\ , \qquad (4.184)$$

these transformations will have the following form, very similar to the $\mathcal{N}=1$ special superconformal one:

$$\delta_\eta \Phi^I| = k\eta_s^A D_A \Phi^I| + 2ikW^I\theta\eta \ . \tag{4.185}$$

However, the boundary condition (4.38) is broken under the second SUSY transformation. To see that, let us calculate its variation:

$$\begin{aligned}
\delta_\eta \Phi^{\hat{I}}| &= \frac{1}{2}e^{kz}\bar{D}^2\left[\Omega^{\hat{I}J}K_J(\theta\eta+\bar{\theta}\bar{\eta})\right]\Big| - 12k\Omega^{\hat{I}J}\left(H_J+\frac{G_J}{3k}\right)\Big|\theta\eta - k\eta_s^A D_A\Phi^{\hat{I}}| \\
&= \frac{1}{2}e^{kz}\bar{D}^2\left[\Omega^{\hat{I}J}K_J(\theta\eta+\bar{\theta}\bar{\eta})\right]\Big| - 4iX^{\hat{I}}|\theta\eta \ .
\end{aligned} \tag{4.186}$$

Although the first term is controlled by the warp factor, the second term is not, unless we restrict

$$X^{\hat{I}}| = 0 \ , \tag{4.187}$$

or equivalently $G| = \text{const.}$

The geometric picture is clear. When (4.187) is satisfied, on the boundary, the Killing vector X^i becomes parallel to the sub-manifold $\mathcal{S}$: $(A^{\hat{I}} = c^{\hat{I}})$. Since X^i is a physical quantity. (4.187) is not a constraint on the manifold, but rather a preferred choice of coordinate system. The in-homogenous tri-holomorphic condition guarantees that the Killing vector field X is non-singular everywhere; therefore its integral curves define a coordinate z as $X^i\partial_i = \partial/\partial z$. For instance, on 2-$d$ hyper-Kähler cone, when $X^i_j = -\frac{3}{2}k\delta^i_j$, the preferred coordinate choice is (ζ^1,ζ^2) as in Sec. 4.7.3; $\zeta^1=0$ is the sub-manifold satisfying $X^{\hat{I}}|=0$. On $2n$-d complex plane, when $X^i_j = -\frac{3}{2}k\delta^i_j$, any n-d linear sub-space passing through the origin will satisfy the requirement (4.187).

Condition (4.187) is also natural from a symmetry analysis. According to the algebra closure relation (B.11) in App. B.1, X generates a central charge symmetry[6] of the bulk action, which may or may not be broken by boundary conditions. For $X^{\hat{I}}| \neq 0$, the boundary condition (4.38) is not invariant under this symmetry. Although the AdS isometry group is preserved, the full supergroup $SU(2,2|1)$ is not; thus the second SUSY is broken on the boundary. The situation here is similar to the vector sector we just discussed. And (4.187) is the necessary condition to preserve the central charge symmetry and the second SUSY as well.

Let us discuss the case when (4.187) is satisfied. The breaking effect on $\Phi^{\hat{I}}$ then becomes

$$\delta_\eta \Phi^{\hat{I}}| = \frac{1}{2} e^{kz} \bar{D}^2 \left[\Omega^{\hat{I}J} K_J(\theta\eta + \bar{\theta}\bar{\eta}) \right] \Big| \ , \tag{4.188}$$

which is really controlled by the warp factor. So the boundary condition (4.38) does not break supersymmetry. Therefore n unrestricted superfields Φ^I on the boundary really transform as 4-d $\mathcal{N}=1$ superconformal multiplets.

The off-shell transformations (4.185) can be also produced from the on-shell SUSY transformations on $(A^I|, \chi^I|)$.

$$\delta A^I| \ = \ e^{-\frac{1}{2}kz}\epsilon\chi^I| + ike^{-\frac{1}{2}kz} x^m \delta^a_m \bar{\eta}\bar{\sigma}_a \chi^I| \tag{4.189}$$

$$\begin{aligned} \delta(e^{-\frac{1}{2}kz}\chi^I)| \ &= \ i\delta^m_a \sigma^a \bar{\epsilon}\partial_m A^I| + kx^n \delta^a_n \eta^{am} \sigma_b \bar{\sigma}_a \eta \partial_m A^I| - 2i\eta X^I| \\ &\quad -\epsilon \mathcal{F}^I + ikx^m \delta^a_m \sigma_a \bar{\eta} \mathcal{F}^I \ , \end{aligned} \tag{4.190}$$

[6] As in (2.26).

where induced 4-d auxiliary fields are defined as

$$\begin{aligned}\mathcal{F}^I &\equiv \Omega^{I\hat{J}} g_{\hat{J}\hat{K}^*}(\partial_5 A^{*\hat{K}^*} + i\bar{X}^{\hat{K}^*})| + \frac{1}{2}\Gamma^I_{PQ}\chi^P|\chi^Q| \\ &= \Omega^{I\hat{J}} g_{\hat{J}\hat{K}^*}\partial_5 A^{*\hat{K}^*}| + \frac{1}{2}\Gamma^I_{PQ}\chi^P|\chi^Q| \ . \end{aligned} \quad (4.191)$$

Here we have taken conditions (4.85) and (4.187).

Using the 5-d Dirac equation, transformations on $\mathcal{F}$ can be calculated:

$$\delta\mathcal{F}^I = e^{-\frac{1}{2}kz}\left(i\delta^m_a\bar{\epsilon}\bar{\sigma}^a\partial_m\chi^I - kx^n\delta^b_n\delta^m_a\eta\sigma_b\bar{\sigma}^a\partial_m\chi^I - 2k\eta\chi^I + 2iX^I_J\eta\chi^J\right)\Big| \ . \quad (4.192)$$

Comparing with App. D, we find this is the correct off-shell superconformal transformation, where $X^I|$ generates a 4-d R-symmetry.

It is interesting to note that (4.187) is also needed for the second SUSY invariance of the action.

In general, a SUSY transformation only preserves the bulk action up to a surface term. We can calculate the variation as

$$\begin{aligned}\delta_\eta S &= \int d^5x\partial_5\Big\{\mathrm{e}^{-2kz}\int d^4\theta(\theta\eta+\bar{\theta}\bar{\eta})\Big(-2K - 2\Omega^{ij}H_iK_j - 2\bar{\Omega}^{i^*j^*}\bar{H}_{i^*}K_{j^*}\Big)\Big\} \\ &\quad + \int d^5x\partial_5\Big\{\mathrm{e}^{-3kz}\int d^2\theta 4\Big[\Omega^{ij}G_iH_j - (G - < G >)\Big]\theta\eta + h.c.\Big\} \ . \end{aligned} \quad (4.193)$$

When the condition (4.187) is satisfied, the unitary gauge is axial and we can always go to $G = 0$ case by redefining $H_{\hat{I}}$ while keeping $H_I| = 0$. (4.193) can be simplified then:

$$\delta_\eta S = -2\int d^5x\partial_5\Big\{\mathrm{e}^{-2kz}\int d^4\theta(\theta\eta+\bar{\theta}\bar{\eta})\Big[K + \frac{1}{3k}(-iX^iK_i + i\bar{X}^{j^*}K_{j^*})\Big]\Big\} \ . \quad (4.194)$$

Using the Killing equation, the following relation can be derived:

$$(-iX^i K_i + i\bar{X}^{j^*} K_{j^*}) = -2D^{(X)} + \Upsilon(A) + \bar{\Upsilon}(A^*) \ , \tag{4.195}$$

where Υ is a holomorphic function.

Hence (4.194) can be written as

$$\delta_\eta S = -2 \int d^5x \partial_5 \Big\{ \mathrm{e}^{-2kz} \int d^4\theta (\theta\eta + \bar{\theta}\bar{\eta}) \Big[K - \frac{2}{3k} \mathrm{D}^{(X)} \Big] \Big\} \ . \tag{4.196}$$

When the boundary $z = z_-$ is kept at a finite location, requiring $\delta_\eta S = 0$ produces a strong constraint on the Killing vector X^i:

$$\nabla_j X^i = -\frac{3k}{2} \delta^i_j \tag{4.197}$$

In general it is not true. Therefore we conclude, for non-conformal sigma models in AdS_5, the second SUSY is broken[7] when the boundary $z = z_-$ is kept at a finite location.

Sending $z_- \to -\infty$, we find the integral in (4.196) is no more harmful than the action integral. If we assume the finite energy condition[8], (4.196) can be claimed as 0.

To conclude, we find the boundary condition (4.38) breaks the second SUSY on the boundary when it is kept at a finite location $z = z_-$. As for the real space-time boundary $z_- \to -\infty$, when (4.187) is satisfied, $\mathcal{N} = 2$ SUSY can be preserved

[7] On the other hand, the "conformal" sigma model action is truly invariant under the second SUSY.

[8] Strictly speaking, the finite energy condition should be applied to the Wick rotated Euclidean AdS_5 version, as shown in [47].

and the bulk action is truly invariant under both of them. Furthermore, $n\ \mathcal{N}=1$ superconformal chiral multiplets will be induced on the boundary.

Discussion

We have seen that half of the chiral superfields survive on the boundary when proper boundary conditions are chosen. We also know from the geometric point of view that the possible values of scalar fields span an n-dimension Kähler sub-manifold of the original hyper-Kählerian one. At last, we find the induced SUSY transformation may take superconformal form. It is then natural to ask whether the induced theory on the boundary is a superconformal invariant sigma model.

A superconformal sigma model requires the vector W^I to be homothetic [53]:

$$\nabla_K W^I = -2i\nabla_K X^I = 2i\delta^I_K \ . \tag{4.198}$$

So far the only constraint on X^i is the inhomogenous tri-holomorphic Killing condition (2.23), otherwise X is still kept as general, so we can not assume this is true.

4.10.3 Gauged Hypermultiplets

Just as in the rigid model, the consistency of the variational principle in the gauged case also requires a set of boundary conditions. We do not need to repeat the calculation again. In fact, it is straightforward to see that because neither counterterm carries the fifth derivative, boundary conditions on gauged hypermultiplets should be the same as in the rigid case.

For simplicity, only the $U(1)$ gauge case will be studied here. As far as the gauge multiplet is concerned, there are two choices of boundary condition: one that breaks the gauge symmetry on the boundary, and one that preserves it. We have already shown for both choices, the induced boundary SUSY transformations on the vector multiplet are 4-d superconformal ones.

Let us then study the second SUSY transformations on the hypermultiplets. Obviously, only the first counterterm shows up in transformations, therefore on the boundary $V| = 0$ produces the second SUSY just as in the rigid case: $\delta_\eta \Phi^I_{gauged}| = \delta_\eta \Phi^I_{rigid}|$. As we explained earlier, they are 4-d superconformal transformations on n chiral multiplets.

The physical picture is clear: when $\mathcal{N} = 1$ gauge vector multiplet vanishes on the boundary, the bulk model locally becomes un-gauged. The rest of the degrees of freedom contain n chiral multiplet A^I plus χ, which is the chiral multiplet from the bulk $\mathcal{N} = 2$ vector field.

What if we choose the gauge invariant choice $(\partial_5 V - \chi - \bar{\chi})| = 0$ instead?

In the Wess-Zumino gauge, this formula becomes a set of component constraints. In particular the following one on the auxiliary field F in χ can be seen clearly from the θ-expansion:

$$F| = 0 \ . \tag{4.199}$$

Using the EOM (3.128) of F, (4.199) becomes

$$\mathrm{P}^{(T)}| = 0 \ , \tag{4.200}$$

which can be rewritten in term of the Killing vector T as

$$T^{\hat{I}}| = 0 \ . \tag{4.201}$$

The geometric meaning of (4.201) is also clear: just like the essential Killing vector X, the tri-holomorphic Killing T must also be parallel to the sub-manifold $\mathcal{S}$. In App. C.2, we show all such tri-holomorphic Killing vectors induce a subgroup of the isometry group on $\mathcal{S}$.

Comparing to the rigid case, on the boundary, the 2nd SUSY transformations among the hypermultiplets are:

$$\delta_\eta \Phi^I_{gauged}| = \delta_\eta \Phi^I_{rigid}| + \frac{1}{2}\mathrm{e}^{kz}\Omega^{I\hat{J}}\bar{D}^2[\partial_{\hat{J}}\Gamma_V(\theta\eta + \bar{\theta}\bar{\eta})]| \ . \tag{4.202}$$

The extra V-dependent terms in the induces transformation vanishes when we impose the consistency condition (4.201)

To see this clearly, we work in the Wess-Zumino gauge, where

$$\partial_i \Gamma_V(\theta\eta + \bar{\theta}\bar{\eta}) = -iK_{ij^*}\bar{T}^{j^*}V(\theta\eta + \bar{\theta}\bar{\eta}) \ ; \tag{4.203}$$

therefore

$$\begin{aligned} \delta_\eta \Phi^I_{gauged}| &= \delta_\eta \Phi^I_{rigid}| + \frac{1}{2}\mathrm{e}^{kz}\Omega^{I\hat{J}}\bar{D}^2[-iK_{\hat{J}\hat{L}^*}\bar{T}^{\hat{L}^*}V(\theta\eta + \bar{\theta}\bar{\eta})]| \\ &= \delta_\eta \Phi^I_{rigid}| \end{aligned} \tag{4.204}$$

The result clearly takes the superconformal transformations' form.

It is important to note that (4.201) is a constraint on a geometric quantity. Since $X^{\hat{I}}| = 0$ already restricts our coordinate system choice, in the gauged case (4.201) restricts the set of isometries that we can preserve on the boundary. So not all bulk gauged models can have such gauge preserving boundary condition. On the contrary, all models can have the gauge breaking boundary condition. In that case, the consistency condition after integrating out the auxiliary field D becomes

$$\partial_5(\mathrm{e}^{-2kz}\Sigma)| = -\mathrm{e}^{-2kz}\mathrm{D}^{(T)}| \ , \tag{4.205}$$

which should be viewed as a Neumann condition on Σ, instead of a geometric constraint.

Again, as the rigid case, We can say the degrees of freedom induced there have supercoformal symmetry.

4.11 Conclusions

In this chapter, we systemically studied the boundary problems in the rigid and gauge sigma model.

We have derived the $\mathcal{N} = 1$ invariant boundary conditions for both the vector and hypermultiplet sectors. In the vector case, two choices of boundary conditions have been found. The first type breaks the gauge invariance on the boundary, while the second type preserves the 4-d vector's degree of freedom and the gauge invariance.

These boundary conditions are self-consistent not only off-shell but also on-shell. After using equations of motion to integrate out the auxiliary fields, the on-shell constraints are of mixed Dirichlet-Neumann type.

For hypermultiplets, we choose special coordinates called Darboux coordinates on the target space to solve the boundary conditions in a special way, so that half of chiral superfields are set to constant values on the space-time boundary, while the other half are kept free. This is a geometric reduction on the hyper-Kähler manifold. The result is a chiral theory involving n chiral multiplets on the boundary.

We have worked out several issues that appeared earlier. For instance, we have found the Gibbons-Hawking-York term for the rigid sigma model. It is obtained only after imposing proper boundary conditions on the hypermultiplets. The Y-term explains why a constraint in the component gauged model should be removed. $\mathcal{N}=1$ invariant Y-terms are also studied. They play interesting roles for us to archive other boundary conditions.

In full AdS_5 case, we have derived a consistency condition on hypermultiplets. This constraint shows up after integrating out the auxiliary field $\mathcal{F}$. The physical implication of this condition is clear: It forbids over-determination in the system.

We derived the induced SUSY transformations on the boundary fields. Instead of using near boundary EOM to solve the asymptotic solutions, we took a more straightforward approach to derive the transformations among fields after imposing proper boundary conditions. It seems that both vector multiplets and hypermultiplets lose half of their degrees of freedom on the space-time boundary, while the other half become superconformal multiplets.

We noticed that boundary conditions may be violated by the second SUSY transformation. Fortunately, in the vector case, we can apply the near boundary profiles of fields to show this breaking vanishes when fields reach the space-time boundary. For hypermultiplets, the story becomes trickier and more interesting. We find that SUSY can be preserved by the boundary conditions as long as the central charge symmetry is preserved. The action is then truly invariant under both supersymmetries as well.

Chapter 5

Concluding Remarks

We have completed the mathematical framework for the minimal ($\mathcal{N} = 1$) supersymmetric nonlinear sigma model living in AdS_5. This is just the starting point for everything interesting.

There can be two directions leading from here.

The first direction is to study the SUSY breaking mechanism on a slice of AdS. This region will have boundaries that explicitly break SUSY to $\mathcal{N} = 1$ on them. One brane will be the location for the hidden sector, which breaks $\mathcal{N} = 1$ SUSY to nothing. The information about this breaking will be transmitted to the visible sector on the other brane, by bulk gauge fields or gravity. Some of the low-lying modes of gravity may become a sigma model. As theorists, we can study the mechanism of such reduction: turning the higher dimension gravity into a sigma model on the background along with other fluctuations.

The second research direction is about the warped space itself. In the last two years, there has been lots of work devoted to supersymmetric theories in various curved dimensions [54–57]. Most of this work just discussed cases with 4 supercharges, i.e. $\mathcal{N} = 1$ in 4-d. Despite that most of them do not even have the right space-time signature to be a real world, they are mathematically beautiful. As far as 8 supercharge case, $\mathcal{N} = 2$ models in AdS_4 have been studied, though only in projective superspace. When we compare our results to theirs, we found a lot of similarities among models, but not identities! This is interesting. We still do not know which property is due to the amount of supercharge and dimensionality, which property is due to superspace technique. In order to fully understand it, maybe we should try to reformulate superspace in an alternative way that realizes other isometries manifestly.

Supersymmetry in curved extra dimension is a still relatively new development and we expect this area to be more and more fruitful and intriguing.

Appendix A

Notation

A.1 Spinor Notation

For 2-component spinors, we chose notation as in Wess-Bagger [16].

In Five-dimension

In 5-d, the irreducible representation of the double covering group of $SO(4,1)$ is a 4-component Dirac spinor. Its explicit form in 2-component notation, along with its Dirac conjugate and Majorana conjugate, is

$$\Psi = \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix} \tag{A.1}$$

$$\overline{\Psi} = \Psi^\dagger \gamma_0 = (\chi, \bar{\psi}) \tag{A.2}$$

$$\widetilde{\Psi} = \Psi^T \mathcal{C} = (-\psi, \bar{\chi}) \ . \tag{A.3}$$

We choose the following γ-matrix convention

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix} \ , \quad \gamma^{\hat{5}} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \ , \tag{A.4}$$

and charge conjugate matrix is

$$\begin{pmatrix} -\varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \ . \tag{A.5}$$

The standard Symplectic Majorana spinor in 5-d is a pair of Dirac spinors with the following relation:

$$\widetilde{\Psi}^i = \epsilon_{ij} \overline{\Psi}^j \ . \tag{A.6}$$

For instance, we take Killing spinors as

$$\epsilon_+ \equiv \epsilon^1 = \begin{pmatrix} -\eta \\ \bar{\epsilon} \end{pmatrix} \ , \qquad \epsilon_- \equiv \epsilon^2 = \begin{pmatrix} \epsilon \\ \bar{\eta} \end{pmatrix} \ . \tag{A.7}$$

For the 5-d non-linear sigma model, we choose the fermion field notation as a generalization of symplectic Majorana convention:

$$\Psi^i = \begin{pmatrix} \chi^i \\ -\Omega^i_{j^*} \bar{\chi}^{j^*} \end{pmatrix} \ . \tag{A.8}$$

The following relations are useful

$$g_{ij^*}\overline{\Psi}^{j^*}\Psi^i = 0 \tag{A.9}$$

$$\overline{\Psi}^{j^*}\epsilon_+ = \Omega^{j^*}{}_k\bar{\epsilon}_-\Psi^k \tag{A.10}$$

$$\overline{\Psi}^{j^*}\gamma^M\epsilon_+ = \Omega^{j^*}{}_k\bar{\epsilon}_-\gamma^M\Psi^k \tag{A.11}$$

$$\overline{\Psi}^{j^*}\gamma^M\gamma^N\epsilon_+ = \Omega^{j^*}{}_k\bar{\epsilon}_-\gamma^N\gamma^M\Psi^k \ . \tag{A.12}$$

Another important relation is the Fierz rearrangement formula. For instance, for tri-spinor products, we have the following fundamental identity:

$$(\bar{\Psi}\Theta)\Phi = \frac{1}{4}(\bar{\Psi}\Phi)\Theta - \frac{1}{4}(\bar{\Psi}\gamma_A\Phi)\gamma^A\Theta - \frac{1}{8}(\bar{\Psi}\gamma_{AB}\Phi)\gamma^{AB}\Theta \ . \tag{A.13}$$

One can build up more identities by replacing Ψ with $\gamma_M\Psi$, $\gamma_M\gamma_N\Psi$, etc.

In Six-dimension

Here we only discuss the 4 spacial plus 2 temporal ("4+2") dimensions AdS_5 embedded in. One can easily generalize the following results to its "5+1" cousin. For spinor notations in various dimensions with diverse signatures, we refer readers to [22, 58].

The minimal irreducible spinor representation of the Lorentz group $SO(4,2)$ is the Weyl spinor. Like in 4-d, gamma matrices in 6-d can be chosen as block off-diagonal:

$$\Gamma^a = \begin{pmatrix} 0 & \gamma^a \\ \gamma^a & 0 \end{pmatrix} \ , \ \Gamma^5 = \begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0 \end{pmatrix} \ , \ \Gamma^6 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \tag{A.14}$$

The anti-commutation relation between them is

$$\{\Gamma^A \Gamma^B\} = -2\eta^{AB} \ . \tag{A.15}$$

Their commutator, on the other hand, are all block diagonal:

$$\Gamma^{ab} = \begin{pmatrix} \gamma^{ab} & 0 \\ 0 & \gamma^{ab} \end{pmatrix} \ , \ \Gamma^{a5} = \begin{pmatrix} \gamma^a \gamma^5 & 0 \\ 0 & \gamma^a \gamma^5 \end{pmatrix} \ , \ \Gamma^{a6} \begin{pmatrix} i\gamma^a & 0 \\ 0 & -i\gamma^a \end{pmatrix} \ ,$$

$$\Gamma^{56} = \begin{pmatrix} i\gamma^5 & 0 \\ 0 & -i\gamma^5 \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 & 0 & 0 \\ 0 & -\mathbb{I} & 0 & 0 \\ 0 & 0 & -\mathbb{I} & 0 \\ 0 & 0 & 0 & \mathbb{I} \end{pmatrix} \ . \tag{A.16}$$

We can define 4 by 4 matrices Σ^{AB} as

$$\Gamma^{AB} = \begin{pmatrix} 2\Sigma^{AB} & 0 \\ 0 & 2\overline{\Sigma}^{AB} \end{pmatrix} \tag{A.17}$$

Analogous to 4-d σ^a, these 15 Σ-matrices are the basis of 6-d SUSY algebra.

A.2 Gravity Notation

We choose the following spin-connection notation:

$$\omega_M^{AB} = \frac{1}{2} e^{AN} e^{BK} \left(e_{MC} \partial_{[N} e_{K]}{}^C - e_{NC} \partial_{[K} e_{M]}{}^C - e_{KC} \partial_{[M} e_{N]}{}^C \right) \ . \tag{A.18}$$

In our coordinate system, the only non-zero AdS_5 spin-connection components are

$$\omega_\mu^{a\hat{5}} = k\mathrm{e}^{-kz}\delta_\mu^a \ . \tag{A.19}$$

The covariant derivative on fermions is defined as

$$\mathcal{D}_M\Psi = \partial_M\Psi + \frac{1}{2}\omega_M^{AB}\Sigma_{AB}\Psi \ , \tag{A.20}$$

where

$$\Sigma^{AB} \equiv \frac{1}{2}\gamma^{AB} = \frac{1}{4}[\gamma^A, \gamma^B] \ . \tag{A.21}$$

A.3 Kähler Geometry

A Kähler manifold is a complex manifold with a Hermitian metric and a sympletic structure. The complex condition allows us to define analyticity and anti-analyticity sing the usual pure imaginary number $i = \sqrt{-1}$. The other two conditions restrict the metric g_{ij^*} to be the second derivative of a real scalar function K:

$$g_{ij^*} = \frac{\partial^2}{\partial A^i \partial A^{*j^*}} K(A, A^*) \ . \tag{A.22}$$

A 2-form can be defined as

$$\omega = \frac{i}{2} g_{ij^*} dA^i \wedge dA^{*j^*} \ , \tag{A.23}$$

whose closeness $d\omega = 0$ is guaranteed by the compatibility condition $\nabla_i g_{j^*} = 0$.

For more mathematical aspects on Kähler geometry, we refer readers to the classic textbook by Griffiths and Harris [59]. For its relation to $\mathcal{N} = 1$ supersymmetric nonlinear sigma models, we refer readers to the book by Wess and Bagger [16]. The following formulas are used in the paper:

$$\Gamma^k_{ij} = g^{kl^*} \frac{\partial}{\partial A^i} g_{jl^*} \tag{A.24}$$

$$R_{ijkl^*} = R_{ijkl} = 0 \tag{A.25}$$

$$R_{ij^*kl^*} = g_{ml^*} \frac{\partial}{\partial A^{*j}} \Gamma^m_{ik} = \frac{\partial}{\partial A^i} \frac{\partial}{\partial A^{*j}} g_{kl^*} - g^{mn^*} \frac{\partial}{\partial A^{*j}} g_{ml^*} \frac{\partial}{\partial A^i} g_{kn^*} \ . \tag{A.26}$$

Covariant derivatives on Kähler manifold are defined as:

$$\nabla_i X^j = \frac{\partial X^j}{\partial A^i} + \Gamma^i_{jk} X^k \tag{A.27}$$

$$\nabla_i \Omega^{jk} = \frac{\partial \Omega^{jk}}{\partial A^i} + \Gamma^j_{ip} \Omega^{pk} + \Gamma^j_{iq} \Omega^{jq} \tag{A.28}$$

$$\nabla_k g_{ik^*} = 0 \tag{A.29}$$

$$\mathcal{D}_\mu \chi^i = \partial_\mu \chi^i + \Gamma^i_{jk} \partial_\mu A^j \chi^k \tag{A.30}$$

$$\mathcal{D}_M \Psi^i = \partial_M \Psi^i + \Gamma^i_{jk} \partial_M A^j \Psi^k \ . \tag{A.31}$$

A.4 Hyper-Kähler Geometry

Any $2n$ complex dimensional hyper-Kähler manifold has three complex structures J's. They satisfy the following relation:

$$J^A J^B = -\mathbb{I}\delta^{AB} - \epsilon^{ABC} J^C \ . \tag{A.32}$$

Complex structures are usually written as $4n \times 4n$ matrices. When a hyper-Kähler manifold is realized in $4n$ real dimensions as $\mathcal{M}$, each J will map $\mathcal{M}^* \to \mathcal{M}^*$, s.t.

$$J^2 = -id \ . \tag{A.33}$$

$\mathcal{M}^*$ is the complexified version of $\mathcal{M}$, this complexification is necessary to realize some J as diagonalized matrix.

Sticking with the general relation (A.32) gives us more freedom. But for practical sake, we make a preferred choice so that J^3 is diagonalized. Physically this allows us to assign "handedness" to chiral and anti-chiral superfields. The usual choices for J are as follows:

$$J^1 = \begin{pmatrix} 0 & -i\Omega^i{}_{j^*} \\ i\Omega^{i^*}{}_j & 0 \end{pmatrix} , \ J^2 = \begin{pmatrix} 0 & \Omega^i{}_{j^*} \\ \Omega^{i^*}{}_j & 0 \end{pmatrix} , \ J^3 = \begin{pmatrix} -i\delta^i_j & 0 \\ 0 & \delta^{i^*}_{j^*} \end{pmatrix} \ . \tag{A.34}$$

On a hyper-Kähler manifold, both Ω_{ij} and $\Omega^i{}_{j^*}$ are covariantly constant. But it is the latter property that guarantees the Nijenhuis tensor to be zero, and all three J's to be integrable. As a result, Hyper-Kähler manifold is complex.

Appendix B

Algebra Closure in Detail

B.1 Hypermultiplets

Here we list the algebra closure relations for the rigid nonlinear sigma model in AdS_5. After using fermion equation of motion, we have the following relations:

$$[\delta_1, \delta_2]A^i = \delta_P A^i + \delta_M A^i + \delta_D A^i + \delta_K A^i + \delta_U A^i \tag{B.1}$$

$$[\delta_1, \delta_2]\chi^i = \delta_P \chi^i + \delta_M \chi^i + \delta_D \chi^i + \delta_K \chi^i + \delta_U \chi^i - \delta_{L_M} \chi^i - \delta_{L_K} \chi^i \ , \tag{B.2}$$

where

$$\delta_P A^i = 2\left(\epsilon_1 \sigma^a \bar{\epsilon}_2 - \epsilon_2 \sigma^a \bar{\epsilon}_1\right)\left(-i\delta^\mu_a \partial_\mu\right) A^i \tag{B.3}$$

$$\delta_P \chi^i = 2\left(\epsilon_1 \sigma^a \bar{\epsilon}_2 - \epsilon_2 \sigma^a \bar{\epsilon}_1\right)\left(-i\delta^\mu_a \partial_\mu\right) \chi^i \tag{B.4}$$

$$\delta_M A^i = 2k\left(\epsilon_1 \sigma^{ab} \eta_2 + \bar{\epsilon}_1 \bar{\sigma}^{ab} \bar{\eta}_2 - \epsilon_2 \sigma^{ab} \eta_1 - \bar{\epsilon}_2 \bar{\sigma}^{ab} \bar{\eta}_1\right)\left(\delta_{\mu a} \delta^\nu_b x^\mu \partial_\nu - \delta^\mu_a \delta_{\nu b} x^\nu \partial_\mu\right) A^i \tag{B.5}$$

$$\delta_M \chi^i = 2k\left(\epsilon_1 \sigma^{ab} \eta_2 + \bar{\epsilon}_1 \bar{\sigma}^{ab} \bar{\eta}_2 - \epsilon_2 \sigma^{ab} \eta_1 - \bar{\epsilon}_2 \bar{\sigma}^{ab} \bar{\eta}_1\right)\left(\delta_{\mu a} \delta^\nu_b x^\mu \partial_\nu - \delta^\mu_a \delta_{\nu b} x^\nu \partial_\mu\right) \chi^i \tag{B.6}$$

$$\delta_D A^i = 2\left(\epsilon_1\eta_2 + \bar{\epsilon}_1\bar{\eta}_2 - \epsilon_2\eta_1 - \bar{\epsilon}_2\bar{\eta}_1\right)\left(\partial_5 + kx^\mu\partial_\mu\right)A^i \tag{B.7}$$

$$\delta_D \chi^i = 2\left(\epsilon_1\eta_2 + \bar{\epsilon}_1\bar{\eta}_2 - \epsilon_2\eta_1 - \bar{\epsilon}_2\bar{\eta}_1\right)\left(\partial_5 + kx^\mu\partial_\mu\right)\chi^i \tag{B.8}$$

$$\delta_K A^i = 2\xi_\eta\left(-ie^{2kz}\delta^\mu_a\partial_\mu + 2ike^{2kz}\delta_{\mu a}x^\mu\partial_5 + 2ik^2\delta_{\mu a}x^\mu x^\nu\partial_\nu - ik^2\delta^\mu_a\eta_{\rho\nu}x^\rho x^\nu\partial_\mu\right)A^i \tag{B.9}$$

$$\delta_K \chi^i = 2\xi_\eta\left(-ie^{2kz}\delta^\mu_a\partial_\mu + 2ike^{2kz}\delta_{\mu a}x^\mu\partial_5 + 2ik^2\delta_{\mu a}x^\mu x^\nu\partial_\nu - ik^2\delta^\mu_a\eta_{\rho\nu}x^\rho x^\nu\partial_\mu\right)\chi^i\ , \tag{B.10}$$

where $\xi_\eta \equiv (\eta_1\sigma^a\bar{\eta}_2 - \eta_2\sigma^a\bar{\eta}_1)$.

$$\delta_U A^i = 2i\left(\epsilon_1\eta_2 - \bar{\epsilon}_1\bar{\eta}_2 - \epsilon_2\eta_1 + \bar{\epsilon}_2\bar{\eta}_1\right)X^i \tag{B.11}$$

$$\delta_U \chi^i = 2i\left(\epsilon_1\eta_2 - \bar{\epsilon}_1\bar{\eta}_2 - \epsilon_2\eta_1 + \bar{\epsilon}_2\bar{\eta}_1\right)\left(X^i_j\chi^j + \frac{3}{2}ik\chi^i\right) \tag{B.12}$$

$$\delta_{L_M}\chi^i = -2k\epsilon_1(\eta_2\chi^i) + 2k\eta_2(\epsilon_1\chi^i) \tag{B.13}$$

$$\delta_{L_K}\chi^i = -2ik^2x^\mu\delta^a_\mu(\eta_1\sigma_a\bar{\eta}_2)\chi^i - 4ik^2x^\mu\delta^a_\mu\sigma_a\bar{\eta}_2(\eta_1\chi^i) - 4ke^{kz}\eta_1(\bar{\eta}_2\Omega^i{}_{j^*}\bar{\chi}^{j^*}) - (1\leftrightarrow 2) \tag{B.14}$$

The last two terms in fermion closure can be removed by a compensating Lorentz transformation.

Closing two infinitesimal SUSY transformations (along Killing spinors) will generate a space-time isometry. Although the metric g_{MN} is invariant such isometry, the veilbein $e_M{}^A$ may transform by a local Lorentz rotation. We can use a compensating Lorentz transformation to restore the vielbein:

$$\delta_\xi e^A_M + \delta_L e^a_\mu = -\xi^N\partial_N e_M{}^A - \partial_M\xi^N e_N{}^A + \lambda^{AB}e_M{}^B = 0\ , \tag{B.15}$$

where the parameter λ^{AB} can be solved as

$$\lambda^{AB} = \xi^N \omega_N^{AB} + e^{MB} e^{NA} (\nabla_M \xi_N) \ . \tag{B.16}$$

Such local compensating transformation λ will also induce a rotation on fields with Lorentz indices. For instance, it rotate fermions as follows

$$\delta_L \begin{pmatrix} \chi^i \\ \\ -\Omega^i{}_{j^*} \bar{\chi}^{j^*} \end{pmatrix} = \frac{1}{2} \lambda^{AB} \Sigma_{AB} \begin{pmatrix} \chi^i \\ \\ -\Omega^i{}_{j^*} \bar{\chi}^{j^*} \end{pmatrix} \ . \tag{B.17}$$

In our coordinates, the only non-zero compensating transformation λ are needed for M and K isometries

$$\lambda^{AB}_{(M_{ab})} = -(\delta^A_b \delta^B_a - \delta^A_a \delta^B_b) \ , \tag{B.18}$$

$$\lambda^{ab}_{(K_c)} = k(\delta^a_c \delta^b_\rho x^\rho - \delta^b_c \delta^a_\rho x^\rho) \ , \qquad \lambda^{a\hat{5}}_{(K_c)} = e^{kz} \delta^a_c \ . \tag{B.19}$$

They remove the extra δ_L terms in fermion closure relations.

B.2 Free Yang-Mills Sector

In 4-component symplectic Majorana notation, the algebra closure relations on non-Abelian gauge vectors are:

$$\begin{aligned} [\delta_1, \delta_2] v_M^{(a)} &= -2i(\bar{\epsilon}_{+1} \gamma^N \epsilon_{+2} - \bar{\epsilon}_{+2} \gamma^N \epsilon_{+1}) \partial_N v_M^{(a)} \\ &\quad -2i \partial_M (\bar{\epsilon}_{+1} \gamma^N \epsilon_{+2} - \bar{\epsilon}_{+2} \gamma^N \epsilon_{+1}) v_N^{(a)} \\ &\quad + \partial_M \Lambda^{(a)} - e f^{abc} \Lambda^{(b)} v_M^{(c)} \ , \end{aligned} \tag{B.20}$$

where the first line is space-time isometry; the second line is the tensor transformation on vector fields; and the last line is the gauge transformation with field dependent parameters as:

$$\Lambda^{(a)} = 2i\bar{\epsilon}_{+1}\epsilon_{+2}\Sigma^{(a)} + 2i(\bar{\epsilon}_{+1}\gamma^N\epsilon_{+2})v_N^{(a)} - (1 \leftrightarrow 2) \ . \tag{B.21}$$

Closure on gauge scalars has the following form:

$$\begin{aligned} [\delta_1, \delta_2]\Sigma_M^{(a)} &= -2i(\bar{\epsilon}_{+1}\gamma^N\epsilon_{+2} - \bar{\epsilon}_{+2}\gamma^N\epsilon_{+1})\partial_N\Sigma^{(a)} \\ &\quad -2i\partial_M(\bar{\epsilon}_{+1}\gamma^N\epsilon_{+2} - \bar{\epsilon}_{+2}\gamma^N\epsilon_{+1})ef^{abc}v_M^{(b)}\Sigma^{(c)} \ , \end{aligned} \tag{B.22}$$

where the second line can be rewritten as a gauge transformation:

$$\delta_\Lambda\Sigma^{(a)} = -ef^{abc}\Lambda^{(b)}\Sigma^{(c)} \ . \tag{B.23}$$

After imposing Dirac equations, the closure relations on gauginos become

$$\begin{aligned} [\delta_1, \delta_2]\lambda_+^{(a)} &= -2i(\bar{\epsilon}_{+1}\gamma^N\epsilon_{+2} - \bar{\epsilon}_{+2}\gamma^N\epsilon_{+1})\partial_N\lambda_+^{(a)} \\ &\quad -\frac{i}{2}(\bar{\epsilon}_{+1}\gamma^N\epsilon_{+2} - \bar{\epsilon}_{+2}\gamma^N\epsilon_{+1})\omega_N^{AB}\gamma_{AB}\lambda_+^{(a)} \\ &\quad +\frac{k}{2}(\bar{\epsilon}_{+1}\gamma^{AB}\epsilon_{+2} - \bar{\epsilon}_{+2}\gamma^{AB}\epsilon_{+1})\gamma_{AB}\lambda_+^{(a)} \\ &\quad +3k(\bar{\epsilon}_{+1}\epsilon_{+2} - \bar{\epsilon}_{+2}\epsilon_{+1})\lambda_+^{(a)} \\ &\quad +\delta_\Lambda\lambda_+^{(a)} \ , \end{aligned} \tag{B.24}$$

where the second and the third lines correspond to the compensating Lorentz rotation; the fourth is the central charge transformation; and the gauge transformation in the last line is

$$\delta_\Lambda\lambda_+^{(a)} = -ef^{abc}\Lambda^{(b)}\lambda_+^{(c)} \ . \tag{B.25}$$

Appendix C

Two Group Theory Theorems

C.1 Integration Constant of $[T, X] = 0$

Here we will show, for holomorphic Killing vectors T and X that commute with each other, the following relation is true:

$$g_{ij^*}(X^i \bar{T}^{j^*} - T^i \bar{X}^{j^*}) = ir \ , \tag{C.1}$$

where r is a constant real number.

The constant $r^{(a)}$ must be 0, if such $T^{(a)}$ generate a semi-simple group.

Proof:

First notice the following relation:

$$\begin{aligned}
\partial_k \left[g_{ij^*}(X^i \bar{T}^{j^*} - T^i \bar{X}^{j^*}) \right] &= g_{ij^*} \nabla_k (X^i \bar{T}^{j^*} - T^i \bar{X}^{j^*}) \\
&= g_{ij^*} (\nabla_k X^i \bar{T}^{j^*} - \nabla_k T^i \bar{X}^{j^*}) \ .
\end{aligned}$$

Using the Killing equation, this becomes

$$\begin{aligned}
\partial_k \left[g_{ij^*}(X^i \bar{T}^{j^*} - T^i \bar{X}^{j^*}) \right] &= -g_{kl^*}(\nabla_{j^*} \bar{X}^{l^*} \bar{T}^{j^*} - \nabla_{j^*} \bar{T}^{l^*} \bar{X}^{j^*}) \\
&= -g_{kl^*}(\partial_{j^*} \bar{X}^{l^*} \bar{T}^{j^*} - \partial_{j^*} \bar{T}^{l^*} \bar{X}^{j^*}) \\
&= -g_{kl^*}[X, T]^{l^*} \\
&= 0 \ .
\end{aligned}$$

Thus the quantity $N \equiv g_{ij^*}(X^i \bar{T}^{j^*} - T^i \bar{X}^{j^*})$ is anti-holomorphic.

However, similarly we have $\partial_{q^*} \left[g_{ij^*}(X^i \bar{T}^{j^*} - T^i \bar{X}^{j^*}) \right] = 0$. So N is actually a constant. Since it is manifestly imaginary, we write it as

$$N \equiv g_{ij^*}(X^i \bar{T}^{j^*} - T^i \bar{X}^{j^*}) = ir \ . \tag{C.2}$$

Suppose all holomorphic T commuting with X generate a group with the following commutation relations:

$$[T^{(a)}, T^{(b)}] = -f^{abc} T^{(c)} \ . \tag{C.3}$$

We can study

$$\bar{X}_{j^*} [\bar{T}^{(a)}, \bar{T}^{(b)}]^{j^*} - X_i [\bar{T}^{(a)}, \bar{T}^{(b)}]^i = i f^{abc} r^{(c)} \ . \tag{C.4}$$

The LHS of (C.4) can be reduced to

$$\begin{aligned}
LHS &= \bar{X}_{j^*} \left(\bar{T}^{(a)j^*}_{p^*} \bar{T}^{(b)p^*} - (a \leftrightarrow b) \right) - h.c. \\
&= \bar{X}_{j^*} \left(\nabla_{p^*} \bar{T}^{(a)j^*} \bar{T}^{(b)p^*} - (a \leftrightarrow b) \right) - h.c. \\
&= \nabla_{p^*} \left(X^i T^{(a)}_i \right) \bar{T}^{(b)p^*} - (a \leftrightarrow b) - h.c. \ .
\end{aligned}$$

Now we can use (C.2) to rewrite these terms,

$$LHS \quad = \quad \nabla_{p^*}\left(X_i T^{(a)i}\right)\bar{T}^{(b)p^*} - (a \leftrightarrow b) - h.c. \ .$$

Further using the Killing condition

$$LHS = -T^{(a)i}(\nabla_i \bar{X}_{p^*})\bar{T}^{(b)p^*} - T^{(b)i}(\nabla_i \bar{X}_{p^*})\bar{T}^{(a)p^*} - (a \leftrightarrow b) = 0 \ . \qquad \text{(C.5)}$$

Thus

$$RHS = -f^{abc} r^c = 0 \ . \qquad \text{(C.6)}$$

$$\Longrightarrow \sum_{a,b} f^{bad} f^{abc} r^{(c)} = \mathrm{tr}(T^{(d)}T^{(c)}) r^{(c)} = \lambda^{(c)} \cdot r^{(c)} = 0 \ . \qquad \text{(C.7)}$$

Note: the last formula is not a summation.

Becasue $T^{(c)}$ belongs to the semi-simple sub-group $\{T^{(A)}\}$, $\lambda^{(c)}$ is non-zero, we then have

$$r^{(c)} = 0 \ . \qquad \text{(C.8)}$$

However, for any $U(1)$ factor in the group, r is still a unfixed constant.

C.2 Induced Isometry Group on Kähler Sub-manifold

We want to show, for any isometry group $\{T^{(a)}\}$ on the hyper-Kähler manifold with the following relation

$$T^{(a)\hat{I}}| = 0 \ ,$$

all the $\{T^{(a)}|\}$ form an isometry group on the Kähler sub-manifold $\mathcal{S} : (A^I; A^{\hat{J}} = c^{\hat{J}})$. The symbol $|$ means evaluating on $\mathcal{S}$.

Proof:

All such $T^{(a)}$ are $2n$-dimension Killing vectors, it is also straightforward to show their n-dimension projections satisfy Killing equations on $\mathcal{S}$ with the induced metric. The only non-trivial question is whether commutators among them close.

First, let us consider the case when $\{T^{(a)}\}$ is semi-simple. The following commutation relation is inherited from the hyperKähler manifold:

$$T^{(a)j}|\partial_j T^{(b)i}| - T^{(b)j}|\partial_j T^{(a)i}| = -f^{abc}T^{(c)i}| \ . \tag{C.9}$$

Obviously

$$\begin{aligned} -f^{abc}T^{(c)\hat{I}}| &= T^{(a)J}|\partial_J T^{(b)\hat{I}}| - T^{(b)J}|\partial_J T^{(a)\hat{I}}| + T^{(a)\hat{J}}|\partial_{\hat{J}} T^{(b)\hat{I}}| - T^{(b)\hat{J}}|\partial_{\hat{J}} T^{(a)\hat{I}}| \\ &= 0 \ . \end{aligned} \tag{C.10}$$

Thus $T^{(c)\hat{I}}| = 0$.

Furthermore

$$\begin{aligned} -f^{abc}T^{(c)I}| &= T^{(a)J}|\partial_J T^{(b)I}| - T^{(b)J}|\partial_J T^{(a)I}| + T^{(a)\hat{J}}|\partial_{\hat{J}} T^{(b)I}| - T^{(b)\hat{J}}|\partial_{\hat{J}} T^{(a)}| \\ &= T^{(a)J}|\partial_J T^{(b)I}| - T^{(b)J}|\partial_J T^{(a)I}| \ . \end{aligned} \tag{C.11}$$

Thus a semi-simple isometry group is induced. Finally, for any $U(1)$ factor T, its induced version $T|$ can be directly add to the induced group as a $U(1)$ factor. This completes the proof.

Appendix D

4-d $\mathcal{N} = 1$ Superconformal Sigma Model

In flat 4-d, the conformal Killing equation is

$$\nabla_\mu \eta_- - \frac{1}{2}\gamma_\mu \eta_+ = 0 \; . \tag{D.1}$$

There are two independent chiral solutions:

$$\eta^1_- = \begin{pmatrix} \epsilon \\ 0 \end{pmatrix} \; , \qquad \eta^1_+ = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \; . \tag{D.2}$$

$$\eta^2_- = \begin{pmatrix} \frac{1}{2}x^i \sigma_\mu \bar{\eta} \\ 0 \end{pmatrix} \; , \qquad \eta^2_+ = \begin{pmatrix} 0 \\ \bar{\eta} \end{pmatrix} \; . \tag{D.3}$$

D.1 On-shell Formalism

On-shell ansätz can be constructed as follows:

$$\begin{aligned}
\delta_\epsilon A^i &= \epsilon\psi^i \\
\delta_\epsilon \psi^i &= i\sigma^\mu\bar{\epsilon}\partial_\mu A^i + iX^i\epsilon - \Gamma^i_{jk}\delta_\epsilon A^j\psi^k
\end{aligned} \tag{D.4}$$

$$\begin{aligned}
\delta_\eta A^i &= ix^\mu\left(\bar{\eta}\bar{\sigma}_\mu\psi^i\right) \\
\delta_\eta \psi^i &= \sigma^\mu\bar{\sigma}^\nu\eta x_\nu\partial_\mu A^i + x^\mu\sigma_\mu\bar{\eta}X^i + i\eta Y^i - \Gamma^i_{jk}\delta_\eta A^j\psi^k \ .
\end{aligned} \tag{D.5}$$

The closure on the bosons is

$$\begin{aligned}
[\delta_\epsilon, \delta_\eta]A^i &= (\epsilon\sigma^{\mu\nu}\eta + \bar{\epsilon}\bar{\sigma}^{\mu\nu}\bar{\eta})(x_\mu\partial_\nu - x_\nu\partial_\mu)A^i \\
&\quad +(\epsilon\eta + \bar{\epsilon}\bar{\eta})(x^\mu\partial_\mu A^i - \frac{i}{2}Y^i) + i(\epsilon\eta - \bar{\epsilon}\bar{\eta})(-\frac{1}{2}Y^i) \ .
\end{aligned} \tag{D.6}$$

We see there is a connection between R-charge and scaling dimension:

$$\Delta = 3R \ . \tag{D.7}$$

Closure on fermion fields determines the following Dirac equation:

$$i\bar{\sigma}^\mu\partial_\mu\psi^i + i\Gamma^i_{jk}\partial_\mu A^j\bar{\sigma}^\mu\psi^k - iX^i_{j^*}\psi^{j^*} - \frac{1}{2}g^{is^*}R_{jr^*ks^*}(\psi^j\psi^k)\bar{\psi}^{r^*} = 0 \ . \tag{D.8}$$

Imposing it, closure on fermions has the following form:

$$[\delta_\epsilon, \delta_\eta]\psi^i = \delta_M\psi^i - \delta_L\psi^i + 2(\bar{\epsilon}\bar{\eta})\psi^i + (-\delta^i_j - i\Gamma^i_{jk}Y^k)\epsilon(\eta\psi^j) + (\delta^i_j + iY^i_j)\eta(\epsilon\psi^j) \ , \tag{D.9}$$

where δ_L is the compensating Lorentz rotation.

Comparing boson and fermion closure requires a constraint:

$$\nabla_j Y^i = 2i\delta^i_j \ . \tag{D.10}$$

This is homethetic Killing condition.

There is another constraint on the vector X too

$$\nabla_i X^j = 0 \ . \tag{D.11}$$

After imposing these, algebra closes on the fields as follows

$$[\delta_\epsilon, \delta_\eta] = \delta_M + \delta_D + \delta_U - \delta_L \ , \tag{D.12}$$

where

$$\delta_D A^i = \xi \left(x^\mu \partial_\mu A^i - \frac{i}{2} Y^i \right) \tag{D.13}$$

$$\delta_D \psi^i = \xi \left(x^\mu \partial_\mu \psi^i + \frac{3}{2}\psi^i + \frac{i}{2}\Gamma^i_{jk} Y^k \psi^j \right) = \xi \left(x^\mu \partial_\mu \psi^i - \psi^i - \frac{i}{2} Y^i_j \psi^j \right) \tag{D.14}$$

$$\delta_U A^i = \zeta \left(\frac{1}{2} Y^i \right) \tag{D.15}$$

$$\delta_U \psi^i = \zeta \left(\frac{i}{2}\psi^i + \frac{1}{2}\Gamma^i_{jk} Y^j \psi^k \right) = \zeta \left(\frac{i}{2}\psi^i - \frac{1}{2} Y^i_j \psi^j \right) \ . \tag{D.16}$$

D.2 Off-shell Formalism

In this section we show how the homethetic Killing condition disappears in the off-shell formalism and how it shows up on-shell.

We can derive the off-shell transformation via $\mathcal{N}=1$ superspace:

$$\delta A^i \;=\; \epsilon\psi^i + ix^\mu(\bar\eta\bar\sigma_\mu\psi^i) \tag{D.17}$$

$$\delta\psi^i \;=\; i\sigma^\mu\bar\epsilon\partial_\mu A^i + (\eta\sigma^\mu\bar\sigma^\nu)x_\mu\partial_\nu A^i + \epsilon\mathcal{F}^i + ix^\mu(\bar\eta\bar\sigma_\mu)\mathcal{F}^i + i\eta Y^i \tag{D.18}$$

$$\delta\mathcal{F}^i \;=\; i\bar\epsilon\bar\sigma^\mu\partial_\mu\psi^i - x^\nu(\eta\sigma^\nu\bar\sigma^\mu\partial_\mu A^i) - 2\eta\psi^i - iY^i_j\eta\psi^i\ . \tag{D.19}$$

In the off-shell formalism, there is no constraint on Y, and the closure on fields is

$$\begin{aligned}[\delta_\epsilon,\delta_\eta]A^i =&\delta_M A^i + (\epsilon\eta+\bar\epsilon\bar\eta)\left(x^\mu\partial_\mu A^i - \frac{i}{2}Y^i\right)\\ &+ i(\epsilon\eta-\bar\epsilon\bar\eta)\left(-\frac{1}{2}Y^i\right)\end{aligned} \tag{D.20}$$

$$\begin{aligned}[\delta_\epsilon,\delta_\eta]\psi^i =&\delta_M\psi^i - \delta_L\psi^i + (\epsilon\eta+\bar\epsilon\bar\eta)\left(x^\mu\partial_\mu\psi^i - \frac{1}{2}\psi^i - \frac{i}{2}Y^i_j\psi^i\right)\\ &+ i(\epsilon\eta-\bar\epsilon\bar\eta)\left(\frac{3i}{2}\psi^i - \frac{1}{2}Y^i_j\psi^j\right)\end{aligned} \tag{D.21}$$

$$\begin{aligned}[\delta_\epsilon,\delta_\eta]\mathcal{F}^i =&\delta_M\mathcal{F}^i + (\epsilon\eta+\bar\epsilon\bar\eta)\left(x^\mu\partial_\mu\mathcal{F}^i - \mathcal{F}^i - \frac{i}{2}Y^i_j\mathcal{F}^i + \frac{i}{4}Y^i_{jp}\psi^j\psi^p\right)\\ &+ i(\epsilon\eta-\bar\epsilon\bar\eta)\left(3i\mathcal{F}^i - \frac{1}{2}Y^i_j\mathcal{F}^j + \frac{1}{4}Y^i_{jp}\psi^j\psi^p\right)\ .\end{aligned} \tag{D.22}$$

In the special case $Y^i_{jp}=0$, (Y^i_j) is the scaling dimension matrix.

So far, we have not made any assumptions on the form of the action. The off-shell transformations have the same form for both the free theory and the coupled theory.

However, if we further assume that the action involves only n chiral multiplets, the form of the action is immediately fixed as a $\mathcal{N}=1$ rigid sigma model, with a target space as Kähler manifold.

We can integrate out the auxiliary field $\mathcal{F}^i$ to compare the off-shell formalism with on-shell one:

$$\mathcal{F}^i = iX^i + \frac{1}{2}\Gamma^i_{jk}\psi^j\psi^k \ . \tag{D.23}$$

Consistency of this EOM under SUSY variation produces the fermion EOM, through x^m-dependent terms:

$$i\bar{\sigma}^\mu\mathcal{D}_\mu\psi^i - iX^i_{r^*}\bar{\psi}^{r^*} - \frac{1}{2}R^i{}_{jr^*k}(\psi^j\psi^k)\bar{\psi}^{r^*} = 0 \ , \tag{D.24}$$

and the following on-shell constraints, through x^m-independent terms:

$$\nabla_j Y^i = 2i\delta^i_j \ , \tag{D.25}$$

$$\nabla_j X^i = 0 \ . \tag{D.26}$$

Because these two constraints are geometric, they must be valid off-shell also!

This example shows how on-shell constraints show up when we specify the action form.

As a conclusion, for n chiral multiplets, if all interactions are among them, the theory must be a nonlinear sigma model. Then conformal invariance requires the target space of these multiplet to admit a homothetic isometry.

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Vita

Jingsheng Li was born in Beijing, P. R. China on Nov 06, 1978. He received a Bachelor of Science degree in the Department of Physics, Peking University in 2001. After that he worked in there as a research assistant studying theoretical nuclear physics before 2004. He then became a Ph.D. student of Johns Hopkins University, under the supervision of Prof. Jonathan Bagger.